Hogere spin diracoperatoren in twee vectorvariabelen in cliffordanalyse

Higher spin Dirac operators in two vector variables in Clifford analysis

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Chapter 1

Introduction

The aim of this doctoral thesis is to study the construction and some properties of a specific *higher spin Dirac operator* in Clifford analysis. In general, it follows from results in [15, 38] that a higher spin Dirac operator $\mathcal{D}_\lambda$ is the unique (up to a multiplicative constant) elliptic conformally invariant first-order differential operator acting as

$$\mathcal{D}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$$

where $\mathbb{V}_\lambda$ denotes an irreducible representation of $\mathrm{Spin}(m)$ with highest weight $\lambda = (\lambda_1 + \frac{1}{2}, \ldots, \lambda_{n-1} + \frac{1}{2}, \frac{1}{2})$ consisting of integers $\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0$ with $n = \lfloor \frac{m}{2} \rfloor$ and m odd. It has been shown in [73] that $\mathbb{V}_\lambda$ can be realised in Clifford analysis as a vector space of polynomials in several vector variables, called the simplicial monogenic polynomials. In the even-dimensional case $m = 2n$, there are two irreducible $\mathrm{Spin}(m)$-representations $\mathbb{V}^\pm_\lambda$ corresponding to highest weights $\lambda = (\lambda_1 + \frac{1}{2}, \ldots, \lambda_{n-1} + \frac{1}{2}, \pm\frac{1}{2})$; the higher spin Dirac operator now acts as $\mathcal{D}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}^\pm_\lambda) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}^\mp_\lambda)$. For convenience, we work in the odd-dimensional case $m = 2n + 1$; the even-dimensional case is not different conceptually.

To every suitable choice of integers $\lambda_1, \ldots, \lambda_{n-1}$ there corresponds a higher spin Dirac operator $\mathcal{D}_\lambda$; the simplest operator in this sequence is the standard Dirac operator ∂_x. Indeed, in the special case that $\lambda_1 = \ldots = \lambda_{n-1} = 0$, the irreducible $\mathrm{Spin}(m)$-module $\mathbb{V}_\lambda$ is isomorphic to the spinor space $\mathbb{S}$ and $\mathcal{D}_\lambda$ coincides with ∂_x. Therefore, higher spin Dirac operators in Clifford analysis can be seen as a generalisation of the standard Dirac operator, whose function-theorical properties lie at the heart of many results in Clifford analysis, see e.g. [4, 30, 42, 68, 29]. In its turn, the standard Dirac operator can be seen as the generalisation of the Cauchy-Riemann operator in the complex plane $\mathbb{C}$. The Dirac operator ∂_x is the topic of chapter 3, where we also give some well-known properties in Clifford analysis, such as the (monogenic) Fischer decomposition, the Cauchy-Kowalewskai (CK) extension and the fact that the space of its homogeneous polynomial null solutions of a fixed degree forms an irreducible Spin(m)-module.

The above-mentioned specific higher spin Dirac operator that plays the main role in this thesis corresponds to the choice of $\lambda_1 = k$, $\lambda_2 = l$ (with integers $k \geq l$) and $\lambda_3 = \ldots = \lambda_{n-1} = 0$. This operator is denoted by $\mathcal{Q}_{k,l}$ and the irreducible Spin(m)-module $\mathbb{V}_\lambda$ is now isomorphic to $\mathcal{S}_{k,l}$, which is the vector space of simplicial monogenic polynomials in two vector variables. In short, we write

$$\mathcal{Q}_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}); \quad f(x; u, v) \mapsto \mathcal{Q}_{k,l} f(x; u, v).$$

Even though explicit examples of higher spin Dirac operators have been studied before (see e.g. [18, 19, 16, 17]), the case of $\mathcal{Q}_{k,l}$ presents complications that were not yet discovered. This operator will be constructed in chapter 6, some function-theoretical results with respect to $\mathcal{Q}_{k,l}$ will be given in chapter 7 and its homogeneous polynomial null solutions (of a fixed degree) will be discussed in chapter 8. We can reveal right now that the vector space of these null solutions is no longer irreducible with respect to the Spin group, contrary to the case of the standard Dirac operator. It appears that this remains true for other higher spin Dirac operators, which makes the study of these operators complicated. Therefore, an important part of this thesis was to describe the decomposition of this kernel space into Spin(m)-irreducibles summands, which we have labeled by their highest weight. In chapter 8 the decomposition is proved by induction on the integers k and l. The decomposition of the kernel space into these irreducibles can be visualised in an aesthetically pleasing way, which is explained in great detail in chapter 9. At that point in the thesis it is known *how* the vector space of h-homogeneous null solutions of the operator $\mathcal{Q}_{k,l}$ behaves under the action of the Spin group; these irreducible modules have yet to be embedded in the kernel space $\mathrm{Ker}_h \mathcal{Q}_{k,l}$. In chapter 10, we have tried to solve this problem.

Let us give a more detailed overview of the chapters.

In order to better understand higher spin Dirac operators, we discuss in chapter 2 definitions, properties and important results in Clifford analysis and representation theory. Standard references are given by [4, 30, 42] and [40, 49], respectively; a not so standard reference of the latter is [44]. Recently, Clifford analysis has proved to be an elegant language to study results in representation theory, and vice versa, representation theoretical techniques are extremely helpful to obtain a better insight in important Clifford analysis results, see e.g. [27, 9, 65, 66, 18, 19, 16, 17]. In particular, we study representations of the Lie group Spin(m), or equivalently, its complex semisimple Lie algebra $\mathfrak{so}(m, \mathbb{C})$, from both a Clifford analysis and representation theoretical point of view. In addition to the spinor space, we focus on the following Spin(m)-irreducible vector spaces: the space of spherical harmonics, the space of spherical monogenics and their respective generalisations to several vector variables, called the simplicial harmonics and simplicial monogenics (see [73]). Several new results, which will be used throughout this thesis, and can therefore be considered as 'basic notions', are given at the end of this chapter where we study the vector space of homogeneous monogenic polynomials in three vector variables, denoted by $\mathcal{M}_{h,k,l}$. This is a reducible vector space with respect to the Spin group and in section 2.4 we give its decomposition into Spin(m)-irreducibles. Of more importance to null solutions of the operator $\mathcal{Q}_{k,l}$ is one of its subspaces, denoted $\mathcal{M}^s_{h,k,l}$, which is, once again, reducible under the action of the Spin group.

As mentioned before, chapter 3 deals with the standard Dirac operator and its well-known properties in Clifford analysis: basic integral formulae, the CK-extension, the harmonic Fischer decomposition and its refinement, the monogenic Fischer decomposition and the monogenic Fischer decomposition of harmonics. All of these decompositions correspond to certain Howe-dual pairs (see e.g. [9, 47, 48]). Also in this chapter, the conformal invariance, the ellipticity and the surjectivity of higher spin Dirac operators have been addressed (see [65, 15, 38, 61]). Furthermore, the twisted Dirac operator is introduced, which is a useful instrument in the construction of higher spin Dirac operators.

Before we arrive at the construction of the main 'star' of this thesis, i.e. the operator $\mathcal{Q}_{k,l}$, it is both instructive and inspiring to consider other examples of higher spin Dirac operators.

In chapter 4, the Rarita-Schwinger operator $\mathcal{R}_k$ is introduced, which is the operator obtained by putting $\lambda_1 = k$ and $\lambda_2 = \ldots = \lambda_{n-1} = 0$ in the highest weight of $\mathbb{V}_\lambda$. As $\mathcal{Q}_{k,l}$ is the immediate generalisation of this operator (with $\mathcal{Q}_{k,0} = \mathcal{R}_k$), this chapter is crucial. In [18] this operator is studied from the Clifford analysis point of view, while in [19] one uses techniques from algebraic geometry and representation theory. Properties of $\mathcal{R}_k$ are given in recent papers [70, 37]. The basic integral formulae of $\mathcal{R}_k$ have been studied recently in great detail in [31]. The decomposition of the kernel space of $\mathcal{R}_k$ into Spin(m)-irreducible summands can be organised as a full triangle and this triangular visualisation is called the Christmas tree. This chapter is kept short, because the operator $\mathcal{Q}_{k,l}$ generalises the results for $\mathcal{R}_k$ and proofs for the Rarita-Schwinger case can easily be adapted from the proofs for $\mathcal{Q}_{k,l}$ in the later chapters.

It was mentioned before that the proof of the decomposition of $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ goes by induction on k and l. Hence, the case $k = l = 1$ is very important, and the operator $\mathcal{Q}_{1,1}$ can be seen as a higher spin Dirac operator acting on functions with values in the vector space of spinor-valued forms. Generally, such operators act on $\mathbb{V}_\lambda$-valued functions with $\lambda_1 = \ldots = \lambda_j = 1$ and $\lambda_{j+1} = \ldots = \lambda_{n-1} = 0$ with $1 \leq j \leq n-1$. This type of higher spin Dirac operators was studied in [16, 17] using techniques of algebraic geometry. In chapter 5 we study these operators in Clifford analysis: the contruction is given and we explicitly prove the decomposition into Spin(m)-irreducibles of the kernel space of the higher spin Dirac operator in case $j = 2$.

At this point we are ready to construct $\mathcal{Q}_{k,l}$. This starts by generalising the monogenic Fischer decomposition of harmonics in one vector variable to the two variable case. This decomposition gives rise to a set of unique (up to a multiplicative constant) conformally invariant first-order differential operators, and the higher spin Dirac operator $\mathcal{Q}_{k,l}$ is one of them. The other ones are called (dual) twistor operators. We prove that they map null solutions to null solutions of this type of higher spin Dirac operator.

Chapter 7 deals with results of $\mathcal{Q}_{k,l}$ in Clifford analysis, such as the generalised CK-extension and its fundamental solution with respect to $\mathcal{Q}_{k,l}$, which yields the generalised basic integral formulae. The inspiration to formulate the expression of the fundamental solution comes from a theorem in [65], which states that null solutions of a conformally invariant first-order differential operator are invariant under the action of the conformal group.

By means of a dimensional argument, together with the above-mentioned induction principle, we have proved the decomposition of $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ in chapter 8. The induction hypothesis is based on the cases $l = 0$ and $k = l = 1$, which correspond to the higher spin Dirac operators of the previous chapters. The null solutions can be split up into two types, which we have denoted type A and type B. The type A solutions will form precisely the vector space $\mathcal{M}^s_{h,k,l}$ and the type B solutions are obtained through an induction principle. To that end, the study of systems of homogeneous Dirac equations in several variables in [24] and the compatibility conditions imposed on the right-hand sides of these equations in order to make the systems solvable, is essential.

The counterpart (or generalisation) of the Christmas tree, which consists of Spin(m)-irreducible spaces of null solutions of the Rarita-Schwinger operator, is investigated in great detail in chapter 9. Even though the result does not resemble a Christmas tree anymore, full triangles and triangular rings are still present. In the case of $\mathcal{Q}_{k,l}$ also hexagon rings occur in a very elegant way that shows a remarkable connection to the pattern of multiplicities of weights in an irreducible representation of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. This pattern is also discussed in section 2.3.3.

The aim of chapter 10 is to embed these irreducible Spin(m)-modules, labeled by their highest weight only up to this point, in the h-homogeneous kernel space of $\mathcal{Q}_{k,l}$. The inversion operator I_Q with respect to $\mathcal{Q}_{k,l}$ can be seen as a generalisation of the refinement of the Kelvin transform (see e.g. [29]) and is essential in the construction of embedding factors. Indeed, this inversion operator maps solutions to solutions and is thus an ideal candidate for the purpose of this chapter. Unfortunately, we can not use the same embedding factors as in the Rarita-Schwinger case, because some Spin(m)-irreducibles occur with multiplicity higher than one. However, we have stated a formulation of the embedding factors, which has been verified in the case of the Rarita-Schwinger operators; this is an alternative way to embed the modules in the latter case. We have also verified it on the space of monogenics $\mathcal{M}^s_{h,k,l}$ and the case $k = 2$ and $l = 1$. Although we are convinced that the conjecture is correct, trying to prove it leads to calculations that quickly explode. This confirms the believe that our method to study more complicated higher spin Dirac operators has not been the most efficient; the 'weightless' approach of [72] looks much more promising. However, many results can be found in this chapter, which very probably will help to unravel patterns for the underlying structure to the solutions.

Finally, we emphasise that the underlying thesis has been written to not only present a mathematical text with definitions, propositions and proofs, but also to show the, sometimes lengthy, calculations needed to establish our results, and also to provide the reader with a highly self-contained text with insights and formulae that will probably show valuable in future research on this topic.

Chapter 2

Basic notions

In this chapter we gather some important and useful results in Clifford analysis and representation theory, together with recent notions and less well-known results in Clifford analysis that will be used throughout this thesis.

Section 2.1 and section 2.2 are devoted to Clifford algebras and Clifford analysis, respectively. Originally, Clifford analysis is a hypercomplex function theory of functions taking values in a Clifford algebra. It is centred around function-theoretical properties of the Dirac operator, which is an elliptic Spin group invariant (even conformally invariant, see e.g. [38]) first-order differential operator factorising the Laplace operator. In this perspective, Clifford analysis refines the theory of harmonic analysis. For a detailed account of the theory of null solutions of the Dirac operator, so-called Euclidean Clifford analysis, we refer to e.g. [4, 30, 42]. In the more recent branch of Hermitean Clifford analysis, the rotational invariance has been broken by introducing a complex structure on Euclidean space, which reduces the invariance to the unitary group. More on this refinement of Euclidean Clifford analysis can be found in e.g. [5, 6].

Recently, Clifford analysis has proved to be an effective tool to study results in representation theory, and vice versa, techniques from representation theory are good instruments to study Clifford analysis, as it has led to a better insight of famous theorems and results in Clifford analysis, see e.g. [66, 27, 9]. It is therefore important to devote a section in the underlying thesis to representation theory, in particular to representations of (semi)simple complex Lie algebras. Well-known definitions and facts are illustrated with explicit examples in the language of Clifford analysis. For more on (matrix) Lie groups, Lie algebras and representation theory, we refer to e.g. [40, 44, 49].

In section 2.4 finally, vector spaces of polynomial null solutions of the Dirac operator in several variables are introduced. Under the action of the Spin group or its complex Lie algebra $\mathfrak{so}(m, \mathbb{C})$, these vector spaces decompose in irreducible modules, which in their turn correspond to certain vector spaces of polynomials. An overview of these irreducible finite-dimensional representations of the Spin group in Clifford analysis can be found in [73].

2.1 Clifford algebras

2.1.1 Definition of $\mathbb{R}_m$ and $\mathbb{C}_m$

In what follows, we work over the Euclidean space $\mathbb{R}^m$ with orthonormal basis $(e_1, \ldots, e_m)$. Denote by $\mathbb{R}_m$ the real universal Clifford algebra generated by these basis elements satisfying the multiplication relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad 1 \leq i, j \leq m.$$

Let $h \in \mathbb{N}$ and $A = \{i_1, i_2, \ldots, i_h\}$ such that $1 \leq i_1 < i_2 < \cdots < i_h \leq m$. Every element of $\mathbb{R}_m$ is of the form $\sum_A a_A e_A$ with $a_A \in \mathbb{R}$ and $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$. If $A = \emptyset$, we put $e_\emptyset = 1$. Elements of a Clifford algebra are referred to as Clifford numbers. In case $|A| = 1$, we use the term Clifford vectors. Elements of the form $\sum_A a_A e_A$ with $|A| = k$ are called k-vectors; they form a subspace $\mathbb{R}_m^{(k)}$. Hence the following multivector structure of the Clifford algebra:

$$\mathbb{R}_m = \mathbb{R}_m^{(0)} \oplus \mathbb{R}_m^{(1)} \oplus \cdots \oplus \mathbb{R}_m^{(m)}.$$

The space $\mathbb{R}^m$ is embedded in $\mathbb{R}_m$ by identifying $(x_1, \ldots, x_m)$ with the real Clifford vector $x = \sum_{j=1}^m e_j x_j$. The multiplication of two vectors x and y is given by $xy = -\langle x, y \rangle + x \wedge y$ with

$$\langle x, y \rangle = \sum_{j=1}^{m} x_j y_j \quad \text{and} \quad x \wedge y = \sum_{1 \leq i < j \leq m} e_i e_j (x_i y_j - x_j y_i)$$

i.e. the scalar-valued Euclidean inner product and the bivector-valued wedge product, respectively. The norm squared of x is defined as $|x|^2 = \langle x, x \rangle$. The wedge product of a finite number of Clifford vectors in $\mathbb{R}_m$ may also be defined using the Clifford product:

$$x_1 \wedge \ldots \wedge x_k := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma)\, x_{\sigma(1)} \cdots x_{\sigma(k)}$$

with $\mathfrak{S}_k$ the symmetric group on k elements.

The complex Clifford algebra $\mathbb{C}_m$ can be seen as the complexification of $\mathbb{R}_m$, i.e. $\mathbb{C}_m = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}_m$. Every element in $\mathbb{C}_m$ is now of the form $\sum_A a_A e_A$ with $a_A \in \mathbb{C}$.

(Anti-)automorphisms on $\mathbb{C}_m$

Put $e_A = e_{i_1} \cdots e_{i_h}$ in what follows. Three (anti-)automorphisms on $\mathbb{C}_m$ leave the multivector structure invariant:
(i) the *main involution*, which is the automorphism on $\mathbb{C}_m$ defined as

$$\begin{aligned}(a_A e_A)^{\widehat{}} &= (a_A e_{i_1} \cdots e_{i_h})^{\widehat{}} \\ &= a_A(-e_{i_1}) \cdots (-e_{i_h}) = a_A(-1)^h e_A,\end{aligned}$$

(ii) the *reversion*, which is the anti-automorphism on $\mathbb{C}_m$ defined as

$$\begin{aligned}(a_A e_A)^{\widetilde{}} &= (a_A e_{i_1} \cdots e_{i_h})^{\widetilde{}} \\ &= a_A(-e_{i_h}) \cdots (-e_{i_1}) = a_A(-1)^{\frac{h(h-1)}{2}} e_A\end{aligned}$$

and at last,
(iii) the *Hermitean conjugation*, which is the anti-automorphism on $\mathbb{C}_m$ defined as

$$(a_A e_A)^\dagger = \bar{a}_A \overline{e}_A = \bar{a}_A e_{i_h} \cdots e_{i_1} = \bar{a}_A (-1)^{\frac{h(h+1)}{2}} e_A$$

with $\bar{a}_A$ the complex conjugation. We can write the conjugation on $\mathbb{C}_m$ as

$$\overline{e}_A = \widehat{\widetilde{e}}_A = \widetilde{\widehat{e}}_A.$$

The main involution on $\mathbb{C}_m$ has two eigenspaces: the even subalgebra $\mathbb{C}_m^+$ of Clifford numbers e_A with $|A|$ even and the odd subspace $\mathbb{C}_m^-$ of Clifford numbers e_A with $|A|$ odd. Under the identification $e_i e_m \mapsto e_i$ for all $1 \leq i \leq m-1$, it is not difficult to prove that

$$\mathbb{C}_m^+ \cong \mathbb{C}_{m-1}.$$

We also introduce the chirality operator, defined as

$$\theta_m = i^{\frac{m(m+1)}{2}} e_1 \cdots e_m.$$

This operator satisfies $\theta_m^2 = 1$.

Remark 1. *Clifford algebras are examples of superalgebras. A superalgebra over a field $\mathbb{K}$ is a $\mathbb{K}$-module $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ together with a bilinear multiplication such that $\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$, where the subscripts are read modulo 2. This notion can be refined to that of a Lie superalgebra, which is a superalgebra endowed with a Lie superbracket satisfying 'super skew-symmetry' and the 'super Jacobi identity' (see e.g. [39] for more details on superalgebras).*

Subgroups of $\mathbb{R}_m$

The real Clifford algebra $\mathbb{R}_m$ has three important subgroups. The Clifford group $\Gamma(m)$ is defined as the group of Clifford numbers that leave $\mathbb{R}^m$ invariant by means of the following action:

$$\Gamma(m) = \{s \in \mathbb{R}_m \mid sx\hat{s}^{-1} \in \mathbb{R}^m, \forall x \in \mathbb{R}^m\}. \tag{2.1}$$

It contains as a subgroup the Pin group $\mathrm{Pin}(m)$, which is the group consisting of products of any number of unit vectors in $\mathbb{R}^m$, i.e.

$$\mathrm{Pin}(m) = \Big\{ \prod_{j=1}^{k} s_j \mid k \in \mathbb{N},\ s_j \in S^{m-1} \Big\} \subset \Gamma(m)$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$. The Spin group $\mathrm{Spin}(m)$ is the subgroup of $\mathrm{Pin}(m)$ that consists of products of an even number of unit vectors in $\mathbb{R}^m$, i.e.

$$\mathrm{Spin}(m) = \Big\{ \prod_{j=1}^{2k} s_j \mid k \in \mathbb{N},\ s_j \in S^{m-1} \Big\} \subset \mathrm{Pin}(m).$$

Representations of Spin(m)

For the definition of a representation of a Lie group (and more information), we refer to section 2.3.1, or the standard references [40, 49].

The vectorial representation of the Spin group on $\mathbb{R}^m$ is defined as

$$\rho : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{R}^m)$$

with the action on $a \in \mathbb{R}^m$ given by

$$\rho(s)a = sa\bar{s}$$

for all $s \in \text{Spin}(m)$. This defines a rotation in the Euclidean space $\mathbb{R}^m$. Since the map $\rho : \text{Spin}(m) \to SO(m, \mathbb{R})$ is surjective and $\rho(-s) = \rho(s)$, the Spin group is a double cover of the real special orthogonal group $SO(m, \mathbb{R})$.

Another representation of the Spin group on $\mathbb{C}_m$ is defined as

$$\mu : \text{Spin}(m) \to \text{Aut}(\mathbb{C}_m)$$

and the action given by left multiplication on $a \in \mathbb{C}_m$:

$$\mu(s)a = sa$$

for all $s \in \text{Spin}(m)$. This representation is not *irreducible* (see section 2.3.1 or [40, 49]), but one can easily adapt μ to an irreducible representation by restricting $\mathbb{C}_m$ to one of its subspaces. This subspace is the topic of the next section.

2.1.2 Definition of the spinor space

In this section, we will introduce a very important subspace of $\mathbb{C}_m$, called spinor space. As the structure of this space depends heavily on the parity of m (with $m = 2n$ or $m = 2n + 1$), we denote by n in general

$$n = \lfloor \frac{m}{2} \rfloor.$$

Representations of $\mathbb{C}_m$

In case $m = 2n$ even, the algebra $\mathbb{C}_{2n}$ is simple and we have

$$\mathbb{C}_{2n} \cong M_{2^n}(\mathbb{C})$$

where $M_{2^n}(\mathbb{C})$ denotes the vector space of complex $2^n \times 2^n$-matrices. There exists a unique irreducible complex representation, called the space of the Dirac spinors and denoted by $\mathbb{S}_{2n}$. This spinor space $\mathbb{S}_{2n}$ can be thought of as a column in the matrices of $M_{2^n}(\mathbb{C})$. The action of $\mathbb{C}_{2n}$ is left matrix multiplication. Since $\dim M_{2^n}(\mathbb{C}) = 2^{2n}$, it follows hat $\dim \mathbb{S}_{2n} = 2^n$.

Another realisation of the spinor space $\mathbb{S}_{2n}$ is that of a minimal left ideal of $\mathbb{C}_{2n}$, which is explained in one of the next paragraphs.

If $m = 2n + 1$ is odd, the chirality operator θ_{2n+1} belongs to the centre of $\mathbb{C}_{2n+1}$. Hence, this Clifford algebra is not simple, but the direct sum of two mutually annihilating simple ideals:

$$\mathbb{C}_{2n+1} = \left(\frac{1+\theta_{2n+1}}{2}\right)\mathbb{C}_{2n+1} \oplus \left(\frac{1-\theta_{2n+1}}{2}\right)\mathbb{C}_{2n+1}.$$

For reasons that become apparent at the end of the section, we are interested in the odd-dimensional case $m = 2n-1$. It can be shown that

$$\mathbb{C}_{2n-1} \cong M_{2^{n-1}}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C}).$$

There will be two (inequivalent) representations this time, called spaces of Weyl spinors, which can be thought of as columns in the matrices of $M_{2^{n-1}}(\mathbb{C})$. These 2^{n-1}-dimensional spaces, denoted by $\mathbb{S}^{\pm}_{2n}$, will be discussed below.

In order to find a realisation of the spinor space, different from the matrix representation, we introduce a new basis for $\mathbb{C}^m$ consisting of Witt vectors:

$$\mathfrak{f}_j = \frac{e_j - ie_{n+j}}{2}, \quad \mathfrak{f}_j^\dagger = -\frac{e_j + ie_{n+j}}{2} \tag{2.2}$$

with $1 \le j \le n$. If $m = 2n$, they form a basis for $\mathbb{C}^{2n}$. In the odd-dimensional case $m = 2n+1$, one needs to include the element e_m to obtain a basis. The basis elements in (2.2) satisfy the Grassmann identities

$$\{\mathfrak{f}_j, \mathfrak{f}_k\} = \{\mathfrak{f}_j^\dagger, \mathfrak{f}_k^\dagger\} = 0, \ \{\mathfrak{f}_j, \mathfrak{f}_k^\dagger\} = \delta_{jk}$$

with $1 \le j, k \le n$. It can be shown that $\mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ and $\mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$ are Grassmann algebras. In the even-dimensional case $m = 2n$, the primitive idempotent I is defined as the product

$$I = I_1 \cdots I_n \tag{2.3}$$

with the mutually commuting idempotents

$$I_k = \mathfrak{f}_k \mathfrak{f}_k^\dagger = \frac{1 + ie_{n+k}e_k}{2}.$$

for $1 \le k \le n$. If $m = 2n+1$, the primitive idempotent equals

$$I = I_1 \cdots I_n \left(\frac{1+ie_m}{2}\right). \tag{2.4}$$

First, we consider the even-dimensional case $m = 2n$. Using the primitive idempotent, the spinor space $\mathbb{S}_{2n}$ can be realised as a minimal left ideal of $\mathbb{C}_{2n}$:

$$\mathbb{S}_{2n} := \mathbb{C}_{2n} I \cong \mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger) I$$

where the isomorphism follows from the observation that $\mathfrak{f}_j I = 0$ for $1 \leq j \leq n$. Note that, as previously stated, this representation for $\mathbb{S}_{2n}$ is 2^n-dimensional. The action of $\mathbb{C}_{2n}$ is left Clifford multiplication.

Remark 2. *Since $\mathbb{S}_{2n}$ can be seen as one of the 2^n columns in the matrix representation of $\mathbb{C}_{2n}$, we have the following isomorphism (where $\mathbb{C}_{2n}$ and $\mathbb{S}_{2n}$ are considered as vector spaces):*

$$\mathbb{C}_{2n} \cong \underbrace{\mathbb{S}_{2n} \oplus \mathbb{S}_{2n} \oplus \cdots \oplus \mathbb{S}_{2n}}_{2^n \text{ summands}}.$$

Second, in the odd-dimensional case $m = 2n - 1$, we make a distinction between two spaces:

$$\mathbb{S}_{2n}^+ \cong \mathrm{Alg}_{\mathbb{C}}^+(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger) I$$
$$\mathbb{S}_{2n}^- \cong \mathrm{Alg}_{\mathbb{C}}^-(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger) I$$

where $\mathrm{Alg}_{\mathbb{C}}^\pm(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$ denote the even and odd subspace of the Grassmann algebra $\mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$, respectively. By means of the chirality operator θ_{2n}, these (inequivalent) spinor spaces can also be obtained via the decomposition

$$\mathbb{S}_{2n} = \left(\frac{1 + \theta_{2n}}{2}\right) \mathbb{S}_{2n} \oplus \left(\frac{1 - \theta_{2n}}{2}\right) \mathbb{S}_{2n}.$$

If we define $\mathbb{S}_{2n}^\pm = \{u \in \mathbb{S}_{2n} \,|\, \theta_{2n} u = \pm u\}$, then

$$\mathbb{S}_{2n} = \mathbb{S}_{2n}^+ \oplus \mathbb{S}_{2n}^-.$$

It follows that $\dim \mathbb{S}_{2n}^+ = \dim \mathbb{S}_{2n}^- = 2^{n-1}$. It should be noted that $\mathbb{S}_{2n}^\pm$ are subspaces of $\mathbb{C}_{2n}$ and not $\mathbb{C}_{2n-1}$. However, it follows from the isomorphism $\mathbb{C}_{2n-1} \cong \mathbb{C}_{2n}^+$ that there exists a well-defined action of $\mathbb{C}_{2n-1}$ on $\mathbb{S}_{2n}^\pm$.

Complex representations of Spin(m)

Because we have

$$\mathrm{Spin}(2n) \subset \mathbb{R}_{2n}^+ \subset \mathbb{C}_{2n}^+ \cong \mathbb{C}_{2n-1},$$

the two inequivalent irreducible representations of $\mathrm{Spin}(2n)$ are given by the spaces $\mathbb{S}_{2n}^{\pm}$. The odd-dimensional case leads to

$$\mathrm{Spin}(2n+1) \subset \mathbb{R}_{2n+1}^{+} \subset \mathbb{C}_{2n+1}^{+} \cong \mathbb{C}_{2n}$$

in which case there is a unique irreducible representation of $\mathrm{Spin}(2n+1)$, given by

$$\mathbb{S}_{2n} \cong \mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1^{\dagger}, \dots, \mathfrak{f}_n^{\dagger})I$$

where I is the primitive idempotent from (2.4).

Remark 3. *A representation theoretical approach to define the spinor space is discussed in section 2.3.6. In what follows, we work in the odd-dimensional case $m = 2n+1$ and we denote the unique spinor space in short by $\mathbb{S}$. In the even-dimensional case $m = 2n$, we also use the symbol $\mathbb{S}$ to denote the sum $\mathbb{S}_{2n}^{+} \oplus \mathbb{S}_{2n}^{-}$ of the even and odd spinors.*

We end this section with two irreducible complex representations of the Spin group. The representation of the Spin group on k-vectors $\mathbb{C}_m^{(k)}$ is defined as

$$h : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{C}_m^{(k)})$$

with the action on $a \in \mathbb{C}_m^{(k)}$ given by

$$h(s)a = sa\bar{s} \tag{2.5}$$

for all $s \in \mathrm{Spin}(m)$. If m is odd, another important irreducible representation of the Spin group on $\mathbb{C}_m$, which can be seen as a direct sum of isomorphic copies of $\mathbb{S}$ (see remark 2), is defined as

$$l : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{S})$$

with

$$l(s)a = sa \tag{2.6}$$

for $a \in \mathbb{S}$ and $s \in \mathrm{Spin}(m)$.

2.2 Clifford analysis

In this section, we will only touch upon some fundamental definitions. Concepts such as the (refined) Fischer decomposition and the CK-extension will be introduced in chapter 3 and generalised to the case of two vector variables in chapter 6 and chapter 7, respectively.

Let $x = \sum_{j=1}^{m} e_j x_j$ be a real Clifford vector. The Dirac operator on $\mathbb{R}^m$ is defined as

$$\partial_x = \sum_{j=1}^{m} e_j \frac{\partial}{\partial x_j}.$$

It is an elliptic conformally invariant first-order differential operator acting on spinor-valued functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \cong \mathcal{C}^\infty(\mathbb{R}^m) \otimes \mathbb{S}$.

Remark 4. *In case of even dimension m, it suffices to take into account that the Dirac operator changes the parity of the underlying values:*

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^{\pm}_{2n}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^{\mp}_{2n}).$$

In light of remark 3, we can write $\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S})$, which implies that the Dirac operator can be seen as a block diagonal matrix of the form $\left(\begin{array}{c|c} 0 & \cdot \\ \hline \cdot & 0 \end{array}\right)$ where 0 denotes the $2^{n-1} \times 2^{n-1}$ zero matrix.

The Dirac operator can be seen as a generalisation of the Cauchy-Riemann operator in the complex plane $\mathbb{C}$. In Clifford analysis, there also exists a counterpart for the concept of holomorphic functions in the complex plane: a spinor-valued function f is (left) *monogenic* in an open domain $\Omega \subset \mathbb{R}^m$ if and only if it satisfies $\partial_x f = 0$ in Ω. Classical Clifford analysis is centred around the concept of monogeneity (see e.g. [30, 42]). Furthermore, the Dirac operator factorises the Laplace operator in $\mathbb{R}^m$:

$$\Delta_x = -\sum_{j=1}^{m} \frac{\partial^2}{\partial {x_j}^2} = -\partial_x^2$$

which implies that Clifford analysis refines the theory of harmonic analysis.

The traditional Euler operator is given by

$$\mathbb{E}_x = \sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j};$$

it measures the degree of homogeneity of polynomials or polynomial operators. For example,

$$[\mathbb{E}_x, x] = x, \quad [\mathbb{E}_x, \partial_x] = -\partial_x.$$

The Gamma operator, a kind of spherical Dirac operator, is defined as

$$\Gamma_x = -\sum_{i=1}^{m}\sum_{j>i}^{m} e_i e_j (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}) =: -\sum_{i=1}^{m}\sum_{j>i}^{m} e_i e_j L_{ij} \tag{2.7}$$

with L_{ij} the angular momentum operator. The operators $\mathbb{E}_x$ and Γ_x commute. By means of angular coordinates $(r, \omega) \in \mathbb{R} \times S^{m-1}$ with $r = |x|$ and $x = r\omega$, it can be shown that (see e.g. [30, 32])

$$\partial_x = \omega \left(\partial_r + \frac{1}{r}\Gamma_x \right) = \frac{1}{r}\omega(\mathbb{E}_x + \Gamma_x) \tag{2.8}$$

$$\Delta_x = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta_{LB} \tag{2.9}$$

where Δ_{LB} is the (scalar) Laplace-Beltrami operator defined as

$$\Delta_{LB} = \Gamma_x(m - 2 - \Gamma_x). \tag{2.10}$$

Two fundamental identities are

$$\begin{aligned} x\partial_x &= -\mathbb{E}_x - \Gamma_x \\ \partial_x x &= -m - 2\mathbb{E}_x - \partial_x x \Leftrightarrow \{x, \partial_x\} = -m - 2\mathbb{E}_x. \end{aligned}$$

There is an underlying structure to the intertwining relations between the operators x, ∂_x and $\mathbb{E}_x$. This structure was mentioned for the first time in [46] and has been proved in a general context in [27]. See also the paper [9].

Theorem 1. *The operators x and ∂_x generate a finite-dimensional Lie superalgebra, isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$.*

Proof. We have

$$\begin{aligned} [\mathbb{E}_x + \frac{m}{2}, \frac{1}{2}(-x^2)] &= 2\frac{1}{2}(-x^2) \\ [\mathbb{E}_x + \frac{m}{2}, \frac{1}{2}\Delta_x] &= -2\frac{1}{2}\Delta_x \\ [\frac{1}{2}(-x^2), \frac{1}{2}\Delta_x] &= \mathbb{E}_x + \frac{m}{2} \end{aligned}$$

which proves that

$$\mathfrak{g}_0 := \text{span}\left\{ \mathbb{E}_x + \frac{m}{2}, \frac{1}{2}(-x^2), \frac{1}{2}\Delta_x \right\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

(See section 2.3.2 for more on the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.) Furthermore,

$$\begin{aligned} [x, x^2] &= 0 & [\partial_x, x^2] &= 2x & \{x, x\} &= 2x^2 \\ [x, \Delta_x] &= -2\partial_x & [\partial_x, \Delta_x] &= 0 & \{\partial_x, \partial_x\} &= -2\Delta_x^2 \\ [x, \mathbb{E}_x + \frac{m}{2}] &= -x & [\partial_x, \mathbb{E}_x + \frac{m}{2}] &= \partial_x & \{x, \partial_x\} &= -2\big(\mathbb{E}_x + \frac{m}{2}\big). \end{aligned}$$

The anti-commutators imply that

$$\mathfrak{g}_1 := \mathrm{span}\{x, \partial_x\}$$

generates $\mathfrak{g}_0$. It follows from all (anti-)commutators that $\mathfrak{osp}(1|2) \cong \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (see also Remark 1). If we normalise the operators to

$$\frac{1}{2}(-x^2), \frac{1}{2}\Delta_x, \frac{1}{2}\big(\mathbb{E}_x + \frac{m}{2}\big), \frac{1}{2\sqrt{2}}ix, -\frac{1}{2\sqrt{2}}i\partial_x$$

we obtain the standard combinations for $\mathfrak{osp}(1|2)$ (see e.g. [39]). □

Remark 5. *In case of n vector variables $(x_1, \ldots, x_n)$, it follows from a result in [39] that*

$$\mathfrak{osp}(1|2n) \cong \mathrm{Alg}\{x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}\}.$$

We end this section by introducing two important spaces of polynomials in Clifford analysis. Denote by $\mathcal{P}_k(\mathbb{R}^m, V)$ the space of V-valued polynomials of homogeneity degree k ($k \in \mathbb{N}$) fixed and put $\mathcal{P}_0(\mathbb{R}^m, V) = V$. In what follows we will usually choose $V = \mathbb{C}_m$, $V = \mathbb{C}$ or $V = \mathbb{S}$. The *spherical harmonics* of degree k are defined as

$$\mathcal{H}_k = \{f \in \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) \mid \Delta_x f = 0\}$$

and the *spherical monogenics* of degree k are defined as

$$\mathcal{M}_k = \{f \in \mathcal{P}_k(\mathbb{R}^m, \mathbb{S}) \mid \partial_x f = 0\}$$

with $\mathcal{H}_0 = \mathbb{C}$ and $\mathcal{M}_0 = \mathbb{S}$.

The following fact is well-known.

Lemma 1. *The space $\mathcal{M}_k$ contains eigenvectors of the operators $\mathbb{E}_x$ and Γ_x with*

$$\mathbb{E}_x P_k(x) = k P_k(x), \quad \Gamma_x P_k(x) = -k P_k(x)$$

for all $P_k \in \mathcal{M}_k$.

Proof. The operators are simultaneously diagonalisable since $[\mathbb{E}_x, \Gamma_x] = 0$. The statement then follows from the identity $\mathbb{E}_x + \Gamma_x = -x\partial_x$. □

2.3 Representation theory

The aim of this section is to briefly discuss the subject of Lie groups and Lie algebras and, in particular, the notion of a 'highest weight of a representation of a semisimple complex Lie algebra'. The latter requires the introduction of many definitions and constructions but we have tried to keep discussion of these aspects to a minimum. The interested reader is referred to the standard works [40, 49]. We follow roughly the approach from [44], where *matrix Lie groups* instead of *Lie groups* are considered. It can be shown that every matrix Lie group is a Lie group; the advantage of matrix Lie groups is that they can be studied without having to go through (a lot of) manifold theory.

We conclude this section by introducing, from a representation theoretical and Clifford analysis point of view, the vector spaces of *simplicial harmonics* and *simplicial monogenics*, which are examples of irreducible finite-dimensional representations of the (matrix) Lie group $\mathrm{Spin}(m)$.

2.3.1 Lie groups, Lie algebras and representations

Denote by $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$. Let $M_m(\mathbb{K})$ be the vector space of all $m \times m$-matrices with entries in $\mathbb{K}$. Denote by $GL(m, \mathbb{K}) \subset M_m(\mathbb{K})$ the subgroup of invertible matrices.

Lie groups

A *matrix Lie group* G is a closed subgroup of $GL(m, \mathbb{C})$ in the following sense: if any sequence (A_n) of matrices in G converges to some matrix A then either $A \in G$ or $A \in M_m(\mathbb{C}) \setminus GL(m, \mathbb{C})$, where the convergence of A_n to A is defined as the convergence of each entry $(A_n)_{kl}$ to A_{kl} for all $1 \leq k, l \leq m$.

As mentioned before, it can be proved (see [44]) that every matrix Lie group is a *Lie group*, which is standardly defined as a differentiable manifold G that is also a group and such that the group product $G \times G \to G$ and the inverse map $g \to g^{-1}$ $(g \in G)$ are differentiable.

Examples of (matrix) Lie groups are:

1. the general linear group $GL(m, \mathbb{C})$ and its subgroup $GL(m, \mathbb{R})$ of real invertible $m \times m$-matrices

2. the special linear group $SL(m, \mathbb{R}) = \{A \in GL(m, \mathbb{R}) \mid \det A = 1\}$

3. the real orthogonal and special orthogonal group, denoted $O(m, \mathbb{R})$ and $SO(m, \mathbb{R})$, respectively

4. the double cover of $SO(m, \mathbb{R})$, i.e. the Spin group $\mathrm{Spin}(m)$.

Lie algebras

We will define a Lie algebra by means of *matrix exponentials.* Let X be a matrix in $M_m(\mathbb{C})$. In [44] it is proved that the matrix exponential e^X, defined as the power series

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!},$$

is a continuous function of X. For every matrix $X \in M_m(\mathbb{C})$, the exponential e^{tX} is a smooth curve in $M_m(\mathbb{C})$ and

$$X = \left.\frac{d}{dt}\right|_{t=0} e^{tX}. \tag{2.11}$$

Furthermore, it can be proved that every invertible $m \times m$-matrix Z can be expressed as $Z = e^X$ for some $X \in M_m(\mathbb{C})$.

The *Lie algebra* of a matrix Lie group G, denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for all t in $\mathbb{R}$. The *exponential mapping* for G is the map

$$\exp : \mathfrak{g} \to G.$$

Remark 6. *The Lie algebra $\mathfrak{g}$ of any matrix Lie group G is a real vector space and it can be shown that $\mathfrak{g}$ is a real Lie algebra, i.e. a real subalgebra of $M_n(\mathbb{C})$. Real Lie algebras can easily be complexified, see e.g. [44].*

Even if the elements of a matrix Lie group G have complex entries, the Lie algebra $\mathfrak{g}$ of G is not necessarily a complex vector space. A matrix Lie group G is called complex if its Lie algebra $\mathfrak{g}$ is a complex subspace of $M_n(\mathbb{C})$ (i.e. if $iX \in \mathfrak{g}$ for all $X \in \mathfrak{g}$). It can be shown that this is equivalent to the condition that G is a complex submanifold of $GL(n, \mathbb{C})$. In what follows, we will work mainly with real Lie groups.

By means of (2.11), it is not difficult to find the corresponding Lie algebras of the examples above:

1. The (complex) Lie algebra of $GL(m,\mathbb{C})$ is the space of all complex $m \times m$-matrices, denoted $\mathfrak{gl}(m,\mathbb{C}) = M_m(\mathbb{C})$. The Lie algebra of $GL(m,\mathbb{R})$ is the space of all real $m \times m$-matrices, denoted $\mathfrak{gl}(m,\mathbb{R}) = M_m(\mathbb{R})$.

2. The Lie algebra of $SL(m,\mathbb{R})$ is the space of all traceless real $m \times m$-matrices, denoted $\mathfrak{sl}(m,\mathbb{R})$. Similarly, the complex Lie algebra $\mathfrak{sl}(m,\mathbb{C}) \cong \mathfrak{sl}(m,\mathbb{R}) \otimes \mathbb{C}$ is the space of all traceless complex $m \times m$-matrices.

3. The Lie algebra of $O(m,\mathbb{R})$ and of $SO(m,\mathbb{R})$ is the space of all real $m \times m$-matrices X with $X^T = -X$, denoted $\mathfrak{so}(m,\mathbb{R})$.

4. As $\mathrm{Spin}(m)$ is the double cover of $SO(m,\mathbb{R})$, the Lie algebra of $\mathrm{Spin}(m)$ is also $\mathfrak{so}(m,\mathbb{R})$. Using (2.11), we prove that $\mathfrak{so}(m,\mathbb{R}) \cong \mathbb{R}_m^{(2)}$.

   By definition, we have $e^{tX} \in \mathrm{Spin}(m)$ for all $t \in \mathbb{R}$ and $X \in \mathfrak{so}(m,\mathbb{R})$. It follows from the definition of the Spin group that there exist unit vectors $\omega_i(t) \in S^{m-1}$ such that

$$e^{tX} = \omega_1(t)\omega_2(t)\cdots\omega_{2k}(t)$$

   for a certain $k \in \mathbb{N}$. In particular, for $t = 0$, this leads to

$$\begin{array}{c} 1 = \omega_1(0)\cdots\omega_{i-1}(0)\omega_i(0)\omega_{i+1}(0)\cdots\omega_{2k}(0) \\ \Updownarrow \\ \omega_i(0)\omega_{i-1}(0)\cdots\omega_1(0) = (-1)^i\omega_{i+1}(0)\cdots\omega_{2k}(0). \end{array}$$

   Using this result, we have

$$\begin{aligned} X &= \left.\frac{d}{dt}\right|_{t=0} \omega_1(t)\omega_2(t)\cdots\omega_{2k}(t) \\ &= \sum_{i=1}^{2k} \omega_1(0)\cdots\omega_{i-1}(0)\omega_i'(0)\omega_{i+1}(0)\cdots\omega_{2k}(0) \\ &= \sum_{i=1}^{2k} (-1)^i \omega_1(0)\cdots\omega_{i-1}(0)\underbrace{\omega_i'(0)\omega_i(0)}\omega_{i-1}(0)\cdots\omega_1(0) \end{aligned}$$

   and it follows from

$$\omega_i(t)^2 = -1 \;\Rightarrow\; \omega_i'(0)\omega_i(0) + \omega_i'(0)\omega_i(0) \;=\; 0 \;=\; -2\langle\omega_i'(0),\omega_i(0)\rangle$$

that $\omega_i'(0)\omega_i(0) = \omega_i'(0) \wedge \omega_i(0)$ is a bivector. Now if $\omega \in S^{m-1}$ and $j \neq l$, then $\omega e_j e_l \omega = -(\omega e_j \omega)(\omega e_l \omega)$ is also a bivector, which follows from

$$\langle \omega e_j \omega, \omega e_l \omega \rangle = -\frac{1}{2}\omega(e_j e_l + e_l e_j)\omega = \omega^2 \delta_{jl} = 0.$$

This means that $X \in \mathbb{R}_m^{(2)}$. Since $\dim \mathfrak{so}(m, \mathbb{R}) = \frac{m(m-1)}{2} = \dim \mathbb{R}_m^{(2)}$, we conclude that $\mathfrak{so}(m, \mathbb{R}) \cong \mathbb{R}_m^{(2)}$. As a consequence,

$$\mathfrak{so}(m, \mathbb{C}) \cong \mathfrak{so}(m, \mathbb{R}) \otimes \mathbb{C} \cong \mathbb{C}_m^{(2)}$$

as Lie algebras.

Abstract notion of Lie algebras

A finite-dimensional Lie algebra over the field $\mathbb{K}$ is standardly defined as a finite-dimensional vector space $\mathfrak{g}$ over $\mathbb{K}$ together with a map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

with the following properties:

1. $[\cdot, \cdot]$ is bilinear
2. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$
3. (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

It can be proved that

Theorem 2 (Ado [1])**.** *Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(m, \mathbb{R})$. Every finite-dimensional complex Lie algebra is isomorphic to a complex subalgebra of $\mathfrak{gl}(m, \mathbb{C})$.*

As an illustration of the standard definition of a Lie algebra, we present four examples:

1. The space $M_m(\mathbb{R}) = \mathfrak{gl}(m, \mathbb{R})$ is a real Lie algebra with respect to the bracket operation $[A, B] = AB - BA$. The space $M_m(\mathbb{C}) = \mathfrak{gl}(m, \mathbb{C})$ is a complex Lie algebra with respect to the same bracket operation. For example, we have the following realisation of $\mathfrak{gl}(3, \mathbb{C})$ in Clifford analysis:

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathbb{E}_x + \frac{m}{2} & \langle x, \partial_u \rangle & \langle x, \partial_v \rangle \\ \langle u, \partial_x \rangle & \mathbb{E}_u + \frac{m}{2} & \langle u, \partial_v \rangle \\ \langle v, \partial_x \rangle & \langle v, \partial_u \rangle & \mathbb{E}_v + \frac{m}{2} \end{pmatrix} \tag{2.12}$$

with $x, u, v \in \mathbb{R}^m$ and E_{ij} 3×3-matrices with 1 on entry (i,j) and 0 elsewhere. This realisation will be useful in section 2.4.

2. An alternative proof of the Lie algebra isomorphism $\mathfrak{so}(m,\mathbb{R}) \cong \mathbb{R}_m^{(2)}$ goes as follows. A basis for $\mathfrak{so}(m,\mathbb{R})$ is given by $\{E_{ij} - E_{ji} \,|\, 1 \leq i < j \leq m\}$, E_{ij} being a $m \times m$-matrix with 1 on entry (i,j) and 0 elsewhere. The isomorphism goes by identifying $E_{ij} - E_{ji} \in \mathfrak{so}(m,\mathbb{R})$ with $e_i e_j \in \mathbb{R}_m^{(2)}$. It is easily verified that the Lie bracket, which is the regular commutator, is an element of $\mathbb{R}_m^{(2)}$:

$$[e_i e_j, e_k e_l] \;=\; -2\delta_{ik} e_l e_j - 2\delta_{jk} e_i e_l + 2\delta_{il} e_k e_j + 2\delta_{jl} e_i e_k.$$

3. Another model for $\mathfrak{so}(m,\mathbb{R})$ is given by the set $\{L_{ij} | 1 \leq i < j \leq m\}$, with the angular momentum operator L_{ij} from (2.7). This follows from the isomorphism $e_i e_j \mapsto L_{ij}$. These operators can also be obtained by considering the infinitesimal action of an element of $\mathrm{Spin}(m)$, see (2.19) and (2.20).

To end this section, we define for every $X \in \mathfrak{g}$ a linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}_X(Y) = [X, Y].$$

This means that ad: $X \to \mathrm{ad}_X$ can be viewed as a linear map from $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$. It follows from the Jacobi identity that this is a Lie algebra homomorphism, i.e. a linear map respecting the bracket $[\cdot,\cdot]$.

Simple, reductive, semisimple

Denote by $\mathfrak{g}$ a complex Lie algebra. An *ideal* in $\mathfrak{g}$ is a complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with the property that for all $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$, we have $[X, H] \in \mathfrak{h}$. Furthermore,

- $\mathfrak{g}$ is called *simple* if its only ideals are $\mathfrak{g}$ and $\{0\}$ and $\mathfrak{g}$ is not commutative.
- $\mathfrak{g}$ is called *reductive* if it is isomorphic to a direct sum of Lie algebras with no nontrivial ideals.
- $\mathfrak{g}$ is called *semisimple* if it is isomorphic to a direct sum of simple Lie algebras.

A reductive Lie algebra decomposes as a direct sum of a semisimple algebra and a commutative algebra.

It can be shown that

Theorem 3. *Every complex simple Lie algebra is isomorphic to precisely one algebra from the following list:*

1. $\mathfrak{sl}(n+1,\mathbb{C})$, $n \geq 1$
2. $\mathfrak{so}(2n+1,\mathbb{C})$, $n \geq 2$
3. $\mathfrak{sp}(2n,\mathbb{C})$, $n \geq 3$
4. $\mathfrak{so}(2n,\mathbb{C})$, $n \geq 4$
5. *the exceptional Lie algebras.*

The symplectic Lie algebras $\mathfrak{sp}(2n,\mathbb{C}) \subset M_{2n}(\mathbb{C})$ and the exceptional Lie algebras will not be discussed in what follows. We refer to [40, 44, 49] for more information.

Representations

First, let V be a real finite-dimensional vector space and G a matrix Lie group. A finite-dimensional real *representation* (Π, V) of G is a Lie group homomorphism

$$\Pi : G \to GL(V) \cong \mathrm{Aut}(V).$$

If a basis for V is chosen, this representation can also be expressed as a Lie group homomorphism $G \to GL(m,\mathbb{R})$. If V is a complex finite-dimensional vector space, then a finite-dimensional complex representation is a Lie group homomorphism $G \to GL(m,\mathbb{C})$.

Second, we define *complex* representations of both real and complex Lie algebras. If V is a complex finite-dimensional vector space and $\mathfrak{g}$ a real or complex Lie algebra, then a finite-dimensional complex *representation* (π, V) of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V) \cong \mathrm{End}(V).$$

This homomorphism is $\mathbb{R}$-linear if $\mathfrak{g}$ is a real Lie algebra and $\mathbb{C}$-linear in the case that $\mathfrak{g}$ is complex. If a basis for V is chosen, this representation can also be expressed as a homomorphism $\mathfrak{g} \to \mathfrak{gl}(m,\mathbb{C})$. See e.g. [44, 20].

Let (Π, V) denote any representation of a matrix Lie group G and let W be a subspace of V. This subspace is called *invariant* if $\Pi(g)(w) \in W$ for all $w \in W$ and $g \in G$. An invariant subspace W is called *nontrivial* if $W \neq \{0\}$ and $W \neq V$. If a representation has no nontrivial invariant subspaces, it is called *irreducible*. These terms are defined analogously for representations of Lie algebras.

Two basic representations are the so-called *standard representation* and the *adjoint representation*.

1. A matrix Lie group G is by definition a subset of some $GL(m, \mathbb{C})$. The inclusion map of G into $GL(m, \mathbb{C})$ is an irreducible representation of G, called the *standard representation* of G. The same holds for $\mathfrak{g}$. For example, the standard representation of the Lie algebra $\mathfrak{so}(m, \mathbb{C})$ is the one in which $\mathfrak{so}(m, \mathbb{C})$ acts in the usual way on $\mathbb{C}^m$.
2. The Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $X \mapsto \mathrm{ad}_X$ with $\mathrm{ad}_X(Y) = [X, Y]$ is called the *adjoint representation* of $\mathfrak{g}$. The adjoint representation of $\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}$ is simple. The adjoint representation can also be defined for the Lie group G with Lie algebra $\mathfrak{g}$, as the Lie group homomorphism Ad: $G \to GL(\mathfrak{g})$, $A \mapsto \mathrm{Ad}_A$, with $\mathrm{Ad}_A(X) = AXA^{-1}$.

Important representations in Clifford analysis

We give two examples of important irreducible representations of the Spin group in Clifford analysis. Although these representations are not finite-dimensional, they are induced by the finite-dimensional representations (2.5) and (2.6), respectively.

1. The H-representation is defined as
$$H : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{C}_m^{(k)}))$$
with
$$H(s)f(x) = sf(\bar{s}xs)\bar{s} \tag{2.13}$$
for all $s \in \mathrm{Spin}(m)$. It is easy to verify that $H(s_1 s_2) = H(s_1)H(s_2)$ for $s_i \in \mathrm{Spin}(m)$. The H-representation commutes with the Laplace operator Δ_x on $\mathbb{C}_m$-valued functions.

   In case of $\mathbb{C}$-valued functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{C})$, H reduces to
$$H(s)f(x) = f(\bar{s}xs). \tag{2.14}$$

2. The L-representation is defined as

$$L : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}))$$

with

$$L(s)f(x) = sf(\bar{s}xs) \tag{2.15}$$

for all $s \in \mathrm{Spin}(m)$. This representation commutes with the Dirac operator on $\mathbb{S}$-valued functions. We refer to section 3.1.2 for a proof.

The action of $\mathrm{Pin}(m)$ on $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S})$ is defined as follows:

$$L(s)f(x) = sf(\tilde{s}xs) \tag{2.16}$$

for $s \in \mathrm{Pin}(m)$.

Connection between representation Π and π

The above mentioned representations Π and π are connected in the following way: for a representation (Π, V) of a matrix Lie group G, there is a unique representation (π, V) of its Lie algebra $\mathfrak{g}$ such that

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. In particular, the *derived representation* π can be computed as

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\Big|_{t=0}.$$

We illustrate this for the L-representation. The derived representation dL of $\mathfrak{so}(m, \mathbb{R})$ on $\mathbb{S}$-valued functions can be calculated as

$$dL(A)f(x) = \frac{d}{dt}\Big|_{t=0} e^{tA} f\big(e^{-tA}xe^{tA}\big) \tag{2.17}$$

with $A \in \mathfrak{so}(m, \mathbb{R}) \cong \mathbb{R}_m^{(2)}$. Note that

$$\frac{d}{dt}\Big|_{t=0} e^{-tA}xe^{tA} = [x, A]. \tag{2.18}$$

If $A = e_ie_j$ with $1 \leq i \neq j \leq n$, then $[x, e_ie_j] = 2x_je_i - 2x_ie_j$. We have

$$dL(e_ie_j)f(x) = e_ie_jf(x) + 2x_j\frac{\partial}{\partial x_i}f(x) - 2x_i\frac{\partial}{\partial x_j}f(x)$$

from which

$$dL(e_i e_j) = e_i e_j \mathbf{1} - 2L_{ij}. \tag{2.19}$$

The Dirac operator commutes with dL.

It follows from (2.14) and (2.19) that the derived representation dH of $\mathfrak{so}(m, \mathbb{R})$ on $\mathbb{C}$-valued functions satisfies

$$dH(e_i e_j) = -2L_{ij}. \tag{2.20}$$

The Laplace operator commutes with dH. Both representations are so-called infinitesimal representations of L and H and can be found in [30].

Schur's lemma

In order to state Schur's lemma, we introduce the notion of *intertwining maps.* Let Π and Σ be representations of a matrix Lie group G acting on vector spaces V and W. An intertwining map of representations is a linear map $\phi : V \to W$ with the property that

$$\phi(\Pi(g)v) = \Sigma(g)(\phi(v))$$

for all $v \in V$ and all $g \in G$. This means that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "\Pi(g)"'] & W \arrow[d, "\Sigma(g)"] \\
V \arrow[r, "\phi"'] & W
\end{tikzcd}$$

The representations Π and Σ are said to be *equivalent* if V and W are isomorphic. All this is defined analogously for representations of Lie algebras.

In the following lemma, representations (Π, V) or (π, V) are denoted in short by the vector space V.

Lemma 2 (Schur [60])**.**

1. *Let V and W be irreducible (real or complex) representations of a group or Lie algebra and let $\phi : V \to W$ be an intertwining map. Then, either $\phi = 0$ or ϕ is an isomorphism.*

2. *Let V be an irreducible complex representation of a group or Lie algebra and let $\phi : V \to V$ be an intertwining map of V with itself. Then, $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.*

3. *Let V and W be irreducible (real or complex) representations of a group or Lie algebra and let $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then, $\phi_1 = \lambda\phi_2$ for some $\lambda \in \mathbb{C}$.*

Let us now take a closer look at representations of some semisimple complex Lie algebras which will be used in the next chapters: $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{so}(m, \mathbb{C})$. We will also study representations of the Lie group $\mathrm{Spin}(m)$.

2.3.2 Representations of $\mathfrak{sl}(2, \mathbb{C})$

In what follows, all representations of $\mathfrak{sl}(2, \mathbb{C})$ will be finite-dimensional. Recall from the previous section that $\mathfrak{sl}(2, \mathbb{C})$ is the algebra of traceless 2×2-matrices with complex entries. A basis for this Lie algebra is $\{H, X, Y\}$ with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These basis elements have the following commutation relations:

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H.$$

If A, B and C are operators on a finite-dimensional vector space V with relations

$$[A, B] = 2B, \ [A, C] = -2C, \ [B, C] = A,$$

then the linear map $\pi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V)$ satisfying

$$\pi(H) = A, \ \pi(X) = B, \ \pi(Y) = C$$

will be a representation of $\mathfrak{sl}(2, \mathbb{C})$.

Examples in Clifford analysis

1. A classical example in harmonic analysis (see also Theorem 1):

$$\pi(H) = \mathbb{E}_x + \frac{m}{2}, \ \pi(X) = \frac{1}{2}(-x^2), \ \pi(Y) = \frac{1}{2}\Delta_x.$$

2. An example with operators that will be used frequently in this thesis:

$$\pi(H) = \mathbb{E}_x - \mathbb{E}_u, \ \pi(X) = \langle x, \partial_u \rangle, \ \pi(Y) = \langle u, \partial_x \rangle.$$

3. An example in superspace (see [26]). Denote by x_i the commuting and by $\grave{x}_i$ $(i = 1, \ldots, m)$ the anti-commuting vector variables, then

$$\begin{aligned}
\pi(H) &= \mathbb{E} + \frac{M}{2} = \sum_{i=1}^{m} x_i \partial_{x_i} + \sum_{j=1}^{2n} \grave{x}_j \partial_{\grave{x}_j} + \frac{M}{2} \\
\pi(X) &= \frac{R^2}{2} = \frac{1}{2} \sum_{i=1}^{m} x_i^2 - \frac{1}{2} \sum_{j=1}^{n} \grave{x}_{2j-1} \grave{x}_{2j} \\
\pi(Y) &= \frac{\nabla^2}{2} = \frac{1}{2} \sum_{i=1}^{m} \partial_{x_i}^2 - 2 \sum_{j=1}^{n} \partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}}.
\end{aligned}$$

Next, we investigate properties of irreducible representations (π, V) of $\mathfrak{sl}(2, \mathbb{C})$.

Let u be an eigenvector of $\pi(H)$ with eigenvalue $\alpha \in \mathbb{C}$. This means that $\pi(H)\pi(X)u = (\alpha + 2)\pi(X)u$, which implies that either $\pi(X)u = 0$ or $\pi(X)u$ is an eigenvector for $\pi(H)$ with eigenvalue $\alpha + 2$. Similarly, we have $\pi(H)\pi(Y)u = (\alpha - 2)\pi(Y)u$, which implies that either $\pi(Y)u = 0$ or $\pi(Y)u$ is an eigenvector for $\pi(H)$ with eigenvalue $\alpha - 2$.

It can be proved that

Fact 1 (Properties of a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$).

1. *Every eigenvalue of $\pi(H)$ is an integer.*

2. *If v is a nonzero element of V such that $\pi(X)v = 0$ and $\pi(H)v = \lambda v$, then there is a non-negative integer n such that $\lambda = n$. The vectors*

$$v, \pi(Y)v, \ldots, \pi(Y)^n v$$

are linearly independent and their span is an irreducible invariant subspace of dimension $n + 1$.

3. *For each integer $n \geq 0$, there is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ with dimension $n + 1$.*

This can be visualised as follows:

$$\begin{tikzcd} 0 & v \arrow[l, bend left, "\pi(X)"] \arrow[r, shift left, "\pi(Y)"] \arrow[loop below, "\pi(H)"] & \pi(Y)v \arrow[l, shift left, "\pi(X)"] \arrow[r, shift left, "\pi(Y)"] \arrow[loop below, "\pi(H)"] & \cdots \arrow[l, shift left, "\pi(X)"] \arrow[r, shift left, "\pi(Y)"] & \pi(Y)^n v \arrow[l, shift left, "\pi(X)"] \arrow[loop below, "\pi(H)"] \arrow[r, bend left, "\pi(Y)"] & 0 \end{tikzcd}$$

Furthermore,

Proposition 1. *Every finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ decomposes as a direct sum of irreducible invariant subspaces.*

Casimir operator

We introduce the Casimir operator for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, which we will need in chapter 5. In general, the Casimir operator is an element of the center of the universal enveloping algebra of a Lie algebra. For details we refer to e.g. [21, 40, 49]. It can be shown (see e.g. [32]) that the Casimir operator $\mathcal{C}$ for $\mathfrak{sl}(2,\mathbb{C})$ is given by

$$\mathcal{C} = H^2 + 2(XY + YX) \tag{2.21}$$

which can be written as

$$\mathcal{C} = H^2 - 2H + 4XY = H^2 + 2H + 4YX.$$

Fact 2 (Properties of the Casimir operator for $\mathfrak{sl}(2,\mathbb{C})$)**.**

1. *The Casimir operator commutes with X, Y and H.*
2. *It follows from Schur's lemma that the Casimir operator acts by multiplication on an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$. If the representation has dimension $n+1$, this multiplicative constant is given by $n(n+2)$.*

For example, the Casimir operator for the representation $\pi(H) = \mathbb{E}_x + \frac{m}{2}$, $\pi(X) = \frac{1}{2}(-x^2)$, $\pi(Y) = \frac{1}{2}\Delta_x$ is given by

$$\begin{aligned} \mathcal{C} &= (\mathbb{E}_x + \frac{m}{2})(\mathbb{E}_x + \frac{m}{2} - 2) - x^2\Delta_x \\ &= -\Delta_{LB} + \frac{m}{2}(\frac{m}{2} - 2) \end{aligned}$$

where we used the decomposition (2.9).

In what follows, we drop the representation π and refer to the operators $\{\pi(H), \pi(X), \pi(Y)\}$ by $\{H, X, Y\}$.

2.3.3 Representations of $\mathfrak{sl}(3,\mathbb{C})$

Although discussing representations of the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ is more complicated than the discussion of representations of $\mathfrak{sl}(2,\mathbb{C})$, the properties of the previous section will come in handy. It is necessary to introduce several new notions in the case of $\mathfrak{sl}(3,\mathbb{C})$; these notions are generalised in section 2.3.4, where we consider representations of semisimple complex Lie algebras in general.

Consider the following basis for $\mathfrak{sl}(3,\mathbb{C})$, i.e. the space of traceless complex 3×3-matrices:

$$\begin{array}{lll} H_1 = E_{11} - E_{22} & X_1 = E_{12} & Y_1 = E_{21} \\ H_2 = E_{22} - E_{33} & X_2 = E_{23} & Y_2 = E_{32} \\ & X_3 = E_{13} & Y_3 = E_{31}. \end{array}$$

Amongst the commutation relations between these basis elements, we point out that $[H_1, H_2] = 0$ and that $\mathrm{span}\{H_i, X_i, Y_i\}$ $(i = 1, 2)$ is a subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. It is easily verified that

$$\{\mathbb{E}_x - \mathbb{E}_u, \mathbb{E}_u - \mathbb{E}_v, \langle x, \partial_u\rangle, \langle u, \partial_x\rangle, \langle x, \partial_v\rangle, \langle v, \partial_x\rangle, \langle u, \partial_v\rangle, \langle v, \partial_u\rangle\}$$

is a basis for $\mathfrak{sl}(3,\mathbb{C})$ in the language of Clifford analysis. In what follows, all representations of $\mathfrak{sl}(3,\mathbb{C})$ will be finite-dimensional. It can be proved that

Proposition 2. *Every finite-dimensional representation of* $\mathfrak{sl}(3,\mathbb{C})$ *decomposes as a direct sum of irreducible invariant subspaces.*

Weights and roots

Let (π, V) be a representation of $\mathfrak{sl}(3,\mathbb{C})$. Since $[\pi(H_1), \pi(H_2)] = \pi([H_1, H_2]) = 0$, it follows from basic linear algebra that $\pi(H_1)$ and $\pi(H_2)$ can be simultaneously diagonalised. Hence the following definition: an ordered pair $\mu = (m_1, m_2) \in \mathbb{C}^2$ is called a *weight* for π if there exists a $v \neq 0$ in V such that

$$\pi(H_1)v = m_1 v, \;\; \pi(H_2)v = m_2 v;$$

v is called *weight vector* corresponding to μ. The space of all weight vectors v corresponding to μ, including $v = 0$, is the *weight space* of the weight μ. The *multiplicity* of a weight is the dimension of the corresponding weight space.

Every representation π of $\mathfrak{sl}(3,\mathbb{C})$ has at least one weight and can be viewed, by restriction, as a representation of the subalgebra $\operatorname{span}\{H_i, X_i, Y_i\} \cong \mathfrak{sl}(2,\mathbb{C})$ $(i = 1, 2)$. As a result, all of the weights of π are of the form $\mu = (m_1, m_2)$ with m_1 and m_2 integers. These elements μ are called *integral elements.* An ordered pair $\mu = (m_1, m_2)$ with m_1 and m_2 non-negative integers with $m_1 \geq m_2$ is called a *dominant integral element.*

An ordered pair $\alpha = (a_1, a_2) \neq (0, 0) \in \mathbb{C}^2$ is called a *root* if there exists a nonzero $Z_\alpha \in \mathfrak{sl}(3,\mathbb{C})$ such that

$$\mathrm{ad}_{H_1}(Z_\alpha) = [H_1, Z_\alpha] = a_1 Z_\alpha,\ \ \mathrm{ad}_{H_2}(Z_\alpha) = [H_2, Z_\alpha] = a_2 Z_\alpha.$$

The element Z_α is a *root vector* corresponding to the root α. The roots are precisely the nonzero weights of the adjoint representation. The Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ has six roots, which are visualised in Figure 2.1 and listed here:

Z_α	α
X_1	$\alpha_1 := (2, -1)$
X_2	$\alpha_2 := (-1, 2)$
X_3	$\alpha_1 + \alpha_2 = (1, 1)$

Z_α	α
Y_1	$-\alpha_1 = (-2, 1)$
Y_2	$-\alpha_2 = (1, -2)$
Y_3	$-\alpha_1 - \alpha_2 = (-1, -1)$

These six roots form a *root system.* The roots α_1 and α_2 are called *positive simple roots.* All the roots can be expressed as linear combinations of the positive simple roots with integer coefficients and these coefficients are (for each root) either all greater than or equal to zero or all less than or equal to zero.

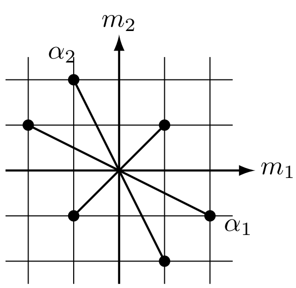


Figure 2.1: Roots of $\mathfrak{sl}(3,\mathbb{C})$ in Euclidean basis.

Highest weight

Let μ_1 and μ_2 be two weights. We say that μ_1 is *higher* than μ_2 (or, equivalently, μ_2 is *lower* than μ_1) if $\mu_1 - \mu_2 = a\alpha_1 + b\alpha_2$ with real $a \geq 0$ and $b \geq 0$. This relation is written as $\mu_1 \succeq \mu_2$ (or $\mu_2 \preceq \mu_1$).

If π is a representation for $\mathfrak{sl}(3, \mathbb{C})$, then a weight μ_0 for π is said to be a *highest weight* if $\mu_0 \succeq \mu$ for all weights μ of π. For example, $\mu_0 = (1, 1)$ is the highest weight for the adjoint representation.

It can be shown that every irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ satisfies the following proposition.

Proposition 3. *Let (π, V) be a representation of $\mathfrak{sl}(3, \mathbb{C})$. If there exists a $v \neq 0$ in V such that*

1. *v is a weight vector with weight μ_0*
2. *$\pi(X_1)v = \pi(X_2)v = 0$*
3. *the smallest invariant subspace of V containing v is all of V,*

the representation π has highest weight μ_0 and its corresponding weight space is one-dimensional.

Fact 3 (Properties of an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$)**.**

1. *Every irreducible representation π of $\mathfrak{sl}(3, \mathbb{C})$ is the direct sum of its weight spaces; i.e., $\pi(H_1)$ and $\pi(H_2)$ are simultaneously diagonalisable in every irreducible representation.*
2. *Every irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ has a unique highest weight and two equivalent representations have the same highest weight.*
3. *The highest weight of an irreducible $\mathfrak{sl}(3, \mathbb{C})$-representation is a dominant integral element and conversely, every dominant integral element occurs as the highest weight of some irreducible representation.*

Three examples

We discuss three examples of irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$ and their highest weights: two so-called fundamental representations and a representation corresponding to the highest weight $(1, 1)$.

The highest weight for the standard representation π_1 of $\mathfrak{sl}(3,\mathbb{C})$ is $(1,0)$. The simultaneous eigenvectors for $\pi_1(H_1) = H_1$ and $\pi_1(H_2) = H_2$ are the basis elements $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$, which have weights $(1,0)$, $(-1,1)$ and $(0,-1)$, respectively.

To construct an irreducible representation π_2 with highest weight $(0,1)$, we modify the standard representation as follows:

$$\pi_2(Z) = -Z^T$$

for all $Z \in \mathfrak{sl}(3,\mathbb{C})$. The simultaneous eigenvectors for $\pi_2(H_1)$ and $\pi_2(H_2)$ are again e_1, e_2 and e_3; this time with respective weights $(-1,0)$, $(1,-1)$ and $(0,1)$. This means that the highest weight for this representation equals $(0,1)$.

Irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ that correspond to highest weights $(1,0)$ or $(0,1)$ are the so-called *fundamental representations*.

Even though an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight $(1,1)$ is isomorphic to the adjoint representation, it is instructive to construct it by means of an alternative approach; this yields techniques that will be useful in section 2.4. Let (π_1, V_1) and (π_2, V_2) be the fundamental representations discussed above. Denote by $v_1 \in V_1$ a weight vector corresponding to the highest weight $(1,0)$ and let $v_2 \in V_2$ be a weight vector with highest weight $(0,1)$; choose $v_1 = e_1$ and $v_2 = e_3$. Now consider the representation π_3 acting on $V_1 \otimes V_2$ as

$$\pi_3 : Z \to \pi_1(Z) \otimes I + I \otimes \pi_2(Z)$$

for all $Z \in \mathfrak{sl}(3,\mathbb{C})$. Consider the vector

$$v_1 \otimes v_2 = e_1 \otimes e_3$$

which is a weight vector with weight $(1,1)$ that satisfies the second condition in Proposition 3. However, the representation π_3 is not irreducible because the smallest invariant subspace W containing the vector $e_1 \otimes e_3$ is not equal to $V_1 \otimes V_2$. This subspace is obtained by starting with $e_1 \otimes e_3$ and applying all possible combinations of Y_1 and Y_2:

$$\begin{aligned}
\pi_3(Y_1)(e_1 \otimes e_3) &= e_2 \otimes e_3 \\
\pi_3(Y_1)^2(e_1 \otimes e_3) &= 0 \\
\pi_3(Y_2)\pi_3(Y_1)(e_1 \otimes e_3) &= e_3 \otimes e_3 - e_2 \otimes e_2
\end{aligned}$$

$$\begin{aligned}
\pi_3(Y_1)\pi_3(Y_2)\pi_3(Y_1)(e_1\otimes e_3) &= e_2\otimes e_1\\
\pi_3(Y_2)^2\pi_3(Y_1)(e_1\otimes e_3) &= -2e_3\otimes e_2\\
&\vdots
\end{aligned}$$

etcetera. A basis for the space spanned by all these vectors is $e_1\otimes e_3$, $e_2\otimes e_3$, $e_1\otimes e_2$, $e_3\otimes e_3-e_2\otimes e_2$, $e_1\otimes e_1-e_2\otimes e_2$, $e_2\otimes e_1$, $e_2\otimes e_1$, $e_3\otimes e_2$ and $e_3\otimes e_1$. The weights for this 8-dimensional representation are $(1,1)$, $(-1,2)$, $(2,-1)$, $(0,0)$, $(1,-2)$, $(-2,1)$ and $(-1,-1)$. Each weight other than $(0,0)$ has multiplicity 1. Because both $e_3\otimes e_3-e_2\otimes e_2$ and $e_1\otimes e_1-e_2\otimes e_2$ are (linearly independent) vectors that correspond to $(0,0)$, this weight has multiplicity 2.

Weights and roots, revisited

In order to discuss an important symmetry of the representations of $\mathfrak{sl}(3,\mathbb{C})$, a basis-independent notion of weight is needed. A fundamental step is the introduction of the *Cartan subalgebra* $\mathfrak{h}$, which is the two-dimensional space of diagonal matrices in $\mathfrak{sl}(3,\mathbb{C})$:

$$\mathfrak{h}=\operatorname{span}\{H_1,H_2\}\subset\mathfrak{sl}(3,\mathbb{C}).$$

This basis-dependent notion will be generalised later to the concept of a maximal abelian subalgebra.

A linear functional $\mu\in\mathfrak{h}^*$ is called a *weight* for a representation (π,V) if there exists a nonzero vector v in V such that

$$\pi(H)v=\mu(H)v$$

for all $H\in\mathfrak{h}$. Such a vector is called a *weight vector* with weight μ.

The important symmetry to the representations of $\mathfrak{sl}(3,\mathbb{C})$ involves the *Weyl group* W, which can be thought of as a group of linear transformations of $\mathfrak{h}$. It can be shown that the Weyl group for $\mathfrak{sl}(3,\mathbb{C})$ is isomorphic to the symmetric group $\mathfrak{S}_3$ and acts by permuting the diagonal entries of elements in $\mathfrak{h}$. We denote this action as $w\cdot H$, for all $H\in\mathfrak{h}$ and $w\in W$. The associated action on the dual space $\mathfrak{h}^*$ is defined as $(w\cdot\mu)(H)=\mu(w^{-1}\cdot H)$ for all $\mu\in\mathfrak{h}^*$ and $w\in W$.

Theorem 4. *Suppose that π is any finite-dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$ and that $\mu\in\mathfrak{h}^*$ is a weight for π. Then, for any $w\in W$, $w\cdot\mu$ is also a weight of π with the same multiplicity as μ.*

This theorem states that the roots are invariant under the action of the Weyl group. In order to visualise this action, it is convenient to identify $\mathfrak{h}^*$ with $\mathfrak{h}$ by means of an inner product on $\mathfrak{h}$ that is invariant under the action of the Weyl group:

$$\langle A, B\rangle := \mathrm{trace}(A^\dagger B) \tag{2.22}$$

with $A^\dagger$ the Hermitean transpose of $A \in \mathfrak{h}$.

We now use this inner product to identify $\mathfrak{h}^*$ with $\mathfrak{h}$. Given any element α of $\mathfrak{h}$, the map $H \to \langle\alpha, H\rangle$ is a linear functional on $\mathfrak{h}$ and an element of $\mathfrak{h}^*$. It follows from the Riesz representation theorem that every linear functional can be represented in this way for a unique α. Under this identification, the action of W on $\mathfrak{h}^*$ coincides with the adjoint action of W on $\mathfrak{h}$. Hence the following basis-independent notion of weight: a nonzero $\alpha \in \mathfrak{h}$ is called a *weight* for π if there exists a nonzero vector v in V such that

$$\pi(H)v = \langle\alpha, H\rangle v \tag{2.23}$$

for all $H \in \mathfrak{h}$. Such a vector is called a *weight vector* with weight α.

Remark 7. *Roots and weights live now in* $\mathfrak{h}$ *instead of* $\mathfrak{h}^*$.

Under the identification $\alpha_1 \leftrightarrow H_1$ and $\alpha_2 \leftrightarrow H_2$, we have $\langle\alpha_1, H_1\rangle = 2$, $\langle\alpha_1, H_2\rangle = -1$, $\langle\alpha_2, H_1\rangle = -1$ and $\langle\alpha_2, H_2\rangle = 2$, which is in agreement with (2.23) and the earlier definition of $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$. Since we have that $||\alpha_1|| = \sqrt{\langle\alpha_1, \alpha_1\rangle} = \sqrt{2}$, $||\alpha_2|| = \sqrt{\langle\alpha_2, \alpha_2\rangle} = \sqrt{2}$ and $\langle\alpha_1, \alpha_2\rangle = -1$, the angle between α_1 and α_2 is $120°$.

The *fundamental weights* μ_1 and μ_2 are defined as

$$\langle\mu_1, H_1\rangle = 1,\ \langle\mu_2, H_1\rangle = 0,\ \langle\mu_1, H_2\rangle = 0,\ \langle\mu_2, H_2\rangle = 1.$$

In terms of the positive simple roots, we have

$$\mu_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \mu_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2;$$

the angle between them is $60°$. The *dominant integral weights* are the linear combinations of μ_1 and μ_2 with non-negative integers coefficients.

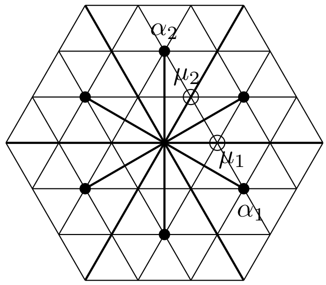


Figure 2.2: Roots of $\mathfrak{sl}(3, \mathbb{C})$ using a Weyl-invariant inner product.

Figure 2.2 provides the same information as Figure 2.1; this time the roots and integral elements are drawn relative to a Weyl-invariant inner product, which yields the triangular grid.

The set $a\mu_1 + b\mu_2$ with real $a \geq 0$ and $b \geq 0$ is called the *closed fundamental Weyl chamber* (relative to the weights μ_1 and μ_2) and is a 60° sector. The dominant integral elements are precisely those integral elements contained in the closed fundamental Weyl chamber. Every highest weight μ_0 lies in the closed fundamental Weyl chamber.

Weyl group

It was mentioned before that the Weyl group acts by permuting the diagonal entries of elements in $\mathfrak{h}$. These actions correspond to the symmetries of an equilateral triangle: the identity, clockwise and counterclockwise rotations by 120°, and three reflections. For example, if $w = (123)$ denotes the cyclic permutation that takes 1 to 2 to 3 to 1, then

$$w \cdot \alpha_1 = \alpha_2, \; w \cdot \alpha_2 = -(\alpha_1 + \alpha_2).$$

This action is a 120° rotation, counterclockwise. In case $w = (12)$, then

$$w \cdot \alpha_1 = -\alpha_1, \; w \cdot \alpha_2 = \alpha_1 + \alpha_2.$$

This corresponds to a reflection about the line perpendicular to α_1.

Remark 8 (An alternative definition of the Weyl group)**.** *The Weyl group can also be defined as a quotient group of Lie groups (see e.g. [44]).*

Weight diagrams

The next theorem states *which* weights occur in a given irreducible $\mathfrak{sl}(3,\mathbb{C})$-representation.

Theorem 5. *Suppose that π is an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight μ_0. An element μ of $\mathfrak{h}$ is a weight of π if and only if the following two conditions are satisfied:*

1. *μ is contained in the convex hull of the orbit of μ_0 under the Weyl group, i.e. the smallest convex subset containing all of the points of the orbit.*

2. *$\mu_0 - \mu$ can be expressed as a linear combination of α_1 and α_2 with integer coefficients.*

Note that the orbit of any $\mu_0 \neq 0$ is a triangle or a hexagon.

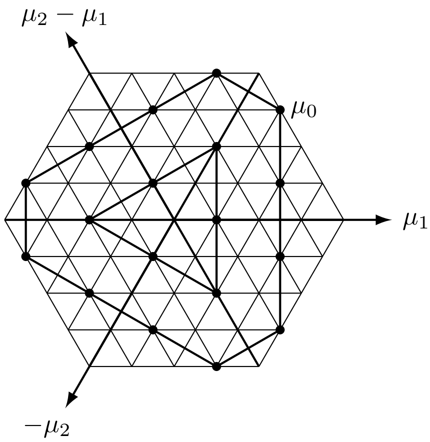


Figure 2.3: Weights of a representation of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight μ_0.

At this point, we know which weights occur but not what the respective multiplicities are. It can be shown (e.g. by means of Theorem 6 in section 2.3.7) that the multiplicities obey the following simple pattern. The weights in the outermost hexagon or triangle have multiplicity 1. The multiplicities then increase by 1 each time one moves inward one hexagon until the you hit triangles, at which point the multiplicities stabilise. Denoting the outermost hexagon 'ring' by h_0, the multiplicity of the weights on h_i, which is the ith hexagon or triangle ring, is given by

$$1 + 2 + \cdots + i = \frac{(i+1)(i+2)}{2};$$

this can be calculated using Kostant's formula (see Theorem 6). Note that this quantity corresponds to a combination of choosing i elements with repetition from 3 elements (corresponding to the action of Y_1, Y_2 and Y_3):

$$\begin{pmatrix} i+3-1 \\ i \end{pmatrix}.$$

2.3.4 Representations of complex semisimple Lie algebras

It was mentioned that the complex simple Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$ have the complete reducibility property. It can be shown that every complex semisimple Lie algebra has this property. Other concepts are generalised as follows.

Cartan subalgebra

The Cartan subalgebra of a complex semisimple Lie algebra is the analogue of $H \in \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{h} \subset \mathfrak{sl}(3,\mathbb{C})$. The *Cartan subalgebra* of a complex semisimple Lie algebra $\mathfrak{g}$ is a complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties:

1. For all H_1 and $H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$.

2. If $[H, X] = 0$ for all $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

3. For all $H \in \mathfrak{h}$, ad_H is diagonalisable.

The Cartan subalgebra $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$. Since $[H_1, H_2] = 0$ implies that $[\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}] = 0$, the operators $\{\mathrm{ad}_H \mid H \in \mathfrak{h}\}$ are simultaneously diagonalisable.

Roots and co-roots

A *root* of $\mathfrak{g}$, relative to the Cartan subalgebra $\mathfrak{h}$, is a nonzero linear functional $\alpha \in \mathfrak{h}^*$ such that there exists a nonzero element X of $\mathfrak{g}$ with

$$\mathrm{ad}_X(H) = [H, X] = \alpha(H)X \tag{2.24}$$

for all $H \in \mathfrak{h}$. The set of all roots is denoted R. Denote by $\mathfrak{g}_\alpha$ the space of all X in $\mathfrak{g}$ satisfying (2.24) for all $H \in \mathfrak{h}$; we call $\mathfrak{g}_\alpha$ a *root space* if α is a root. An element of $\mathfrak{g}_\alpha$ is called a *root vector*. If $\alpha = 0$ then $\mathfrak{g}_0 = \mathfrak{h}$.

Proposition 4 (The Cartan decomposition)**.** *Every complex semisimple Lie algebra* $\mathfrak{g}$ *can be decomposed as the direct sum*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha. \tag{2.25}$$

Fact 4 (Properties of roots)**.**

1. *If α is a root, so is $-\alpha$ and the only multiples of α that are roots are α and $-\alpha$.*

2. *For any X in the root space $\mathfrak{g}_\alpha$, the Hermitean adjoint $X^\dagger$ is in the root space $\mathfrak{g}_{-\alpha}$.*

3. *If α is a root, then the root space $\mathfrak{g}_\alpha$ is one-dimensional.*

4. *For each root α, we can find nonzero elements X_α in $\mathfrak{g}_\alpha$, Y_α in $\mathfrak{g}_{-\alpha}$ and H_α in $\mathfrak{h}$ such that*

$$[H_\alpha, X_\alpha] = 2X_\alpha,\ [H_\alpha, Y_\alpha] = -2Y_\alpha,\ [X_\alpha, Y_\alpha] = H_\alpha.$$

The last property implies that X_α, Y_α and H_α span a subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ with $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$. The elements H_α are unique and are the so-called *co-roots*. Note that $\alpha(H_\alpha) = 2$.

Roots, revisited

Once more, we use the inner product (2.22) to identify $\mathfrak{h}^*$ with $\mathfrak{h}$, which puts roots and co-roots in the same space. It follows from Riesz's theorem that given any linear functional $\alpha \in \mathfrak{h}^*$ (not necessarily a root), there exists a unique element $H^\alpha \in \mathfrak{h}$ such that

$$\alpha(H) = \langle H^\alpha, H \rangle$$

for all $H \in \mathfrak{h}$. It is convenient to permanently identify each root $\alpha \in \mathfrak{h}^*$ with $H^\alpha \in \mathfrak{h}$; we then omit the H^α notation and denote that element of $\mathfrak{h}$ as α.

A *root* of $\mathfrak{g}$ is now a nonzero element α of $\mathfrak{h}$ with the property that there exists a nonzero element $X \in \mathfrak{g}$ with

$$[H, X] = \langle \alpha, H \rangle X \tag{2.26}$$

for all $H \in \mathfrak{h}$. The set of all roots is denoted R. The co-roots satisfy

$$H_\alpha = 2 \frac{\alpha}{\langle \alpha, \alpha \rangle}$$

which is consistent with $\alpha(H_\alpha) = \langle \alpha, H_\alpha \rangle = 2$.

Weyl group

The Weyl group is defined as the set of linear transformations of $\mathfrak{h}$ generated by the reflections w_α ($\alpha \in R$) in the hyperplane $V_\alpha = \{\beta \in \mathfrak{h} \mid \langle H_\alpha, \beta \rangle = 0\}$ with axis the line spanned by β.

Fact 5 (Properties of the Weyl group W)**.**

1. *The inner product (2.22) on $\mathfrak{h}$ is invariant under the action of W.*
2. *The set of roots R is invariant under the action of W.*
3. *The set of co-roots is invariant under the action of W and $w \cdot H_\alpha = H_{w \cdot \alpha}$ for all $w \in W$ and $\alpha \in R$.*

Root systems and positive (simple) roots

In order to define *positive* roots, we introduce the notion of a root system. It can be shown that the roots form a finite set of nonzero elements of a real inner product space E with the following properties:

1. The roots span E.
2. If α is a root, so is $-\alpha$ and the only multiples of α that are roots are α and $-\alpha$.
3. For all roots α and β, $w_\alpha \cdot \beta$ is a root, with w_α defined as before.
4. If α and β are roots, then the quantity $2\langle \alpha, \beta \rangle \langle \alpha, \alpha \rangle^{-1}$ is an integer.

Any collection of vectors in a finite-dimensional real inner product space having these properties is called a *root system.*

Suppose that E is a finite-dimensional real inner product space with $R \subset E$ a root system. A *base* for R is a subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of R such that Δ forms a basis for E as a vector space and such that for each $\alpha \in R$, we have

$$\alpha = n_1\alpha_1 + \cdots + n_r\alpha_r$$

where $n_j \in \mathbb{Z}$ and either $n_j \geq 0$ or $n_j \leq 0$ for all $1 \leq j \leq r$. The roots α for which $n_j \geq 0$ (resp. $n_j \leq 0$) are called *positive* (resp. *negative*) roots with respect to Δ. The elements of Δ are the *positive simple* roots.

Integral and dominant integral elements

An element μ of $\mathfrak{h}$ is called an *integral element* if $\langle \mu, H_\alpha \rangle$ is an integer for each root α. It can be proved that the set of integral elements is invariant under the action of the Weyl group. An element μ of $\mathfrak{h}$ is called an *dominant integral element* if $\langle \mu, H_\alpha \rangle$ is a non-negative integer for each positive simple root α.

The set of $\mu \in \mathfrak{h}$ such that $\langle \mu, H_\alpha \rangle \geq 0$ for all positive simple roots α is called the *closed fundamental Weyl chamber* relative to the given set of positive simple roots. The *open fundamental Weyl chamber* is the set of $\mu \in \mathfrak{h}$ such that $\langle \mu, H_\alpha \rangle > 0$ for all positive simple roots α.

Remark 9 (An alternative definition of the open Weyl chamber)**.** *For each $\alpha \in R$, denote by V_α the hyperplane perpendicular to α. An open Weyl chamber in E (relative to R) is a connected component of the set*

$$E \setminus \bigcup_{\alpha \in R} V_\alpha.$$

It can be shown (see [44]) that the use of the term 'Weyl chamber' in this definition is consistent with its use in the previous definition.

Weights

Suppose π is a finite-dimensional representation of $\mathfrak{g}$ on a vector space V. An element $\mu \in \mathfrak{h}$ is called a *weight* for π if there exists a nonzero vector v in V such that for all $H \in \mathfrak{h}$,

$$\pi(H)v = \langle \mu, H \rangle v.$$

Such a nonzero vector v is called a *weight vector* for μ. The set of all those vectors v, together with $v = 0$, is called the *weight space* with weight μ. The dimension of the weight space is called the *multiplicity* of the weight.

Fact 6 (Properties of weights of representations of $\mathfrak{g}$).

1. *If $\mu \in \mathfrak{h}$ is a weight of some finite-dimensional representation (π, V) of $\mathfrak{g}$, then μ is an integral element.*

2. *Every finite-dimensional representation (π, V) is the direct sum of its weight spaces; i.e., the set of operators of the form $\pi(H)$ $(H \in \mathfrak{h})$ are simultaneously diagonalisable in every finite-dimensional representation.*

3. *For any finite-dimensional representation π of $\mathfrak{g}$, the weights of π and their multiplicity are invariant under the action of the Weyl group.*

Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a base. The *fundamental weights* (relative to Δ) are the elements $\mu_1, \ldots, \mu_r \in \mathfrak{h}$ with the property that

$$\langle \mu_k, H_{\alpha_l} \rangle = \delta_{kl}.$$

They are the weights met along the edges of the closed Weyl chamber.

Highest weight

Let μ_1 and μ_2 be two elements of $\mathfrak{h}$ and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of positive simple roots. Then, μ_1 is *higher* than μ_2 (or, equivalently, μ_2 is *lower* than μ_1) if there exist non-negative real numbers $a_1, \ldots, a_r$ such that $\mu_1 - \mu_2 = a_1\alpha_1 + \cdots + a_r\alpha_r$. This relation is written as $\mu_1 \succeq \mu_2$ or $\mu_2 \preceq \mu_1$.

If π is a representation of $\mathfrak{g}$, then a weight μ_0 for π is said to be a *highest weight* if $\mu_0 \succeq \mu$ for all weights μ of π.

Fact 7 (Properties of irreducible representations of $\mathfrak{g}$).

1. *Every irreducible representation has a highest weight and two irreducible representations with the same highest weight are equivalent.*

2. *The highest weight of an irreducible representation is a dominant integral element and conversely, every dominant integral element occurs as the highest weight of some irreducible representation.*

2.3.5 Representations of $\mathfrak{so}(m,\mathbb{C})$

In this section, we illustrate some definitions of the previous section by investigating finite-dimensional representations of the semisimple Lie algebra $\mathfrak{so}(m,\mathbb{C})$. In particular, we will show that the weights of this algebra occur in two types: integer and half-integer. In section 2.3.6, we discuss some special cases of representations of $\mathfrak{so}(m,\mathbb{C})$ in Clifford analysis.

Recall that $\mathfrak{so}(m,\mathbb{C})$ is the vector space of skew-symmetric complex $m\times m$-matrices with the Lie bracket $[A,B] = AB - BA$. The underlying special orthogonal (real) Lie group $SO(m,\mathbb{R})$ is defined as the group of matrices that leave the inner product $\langle x,y\rangle = x^T y$ invariant, x and y being complex $n\times 1$-matrices. With this definition of orthogonality, there are no nonzero diagonal matrices in the Lie algebra. In case $m = 2n$, the matrices in the Cartan subalgebra look like

$$\begin{pmatrix} 0 & a_1 & & & & & \\ -a_1 & 0 & & & & & \\ & & 0 & a_2 & & & \\ & & -a_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & a_n \\ & & & & & -a_n & 0 \end{pmatrix}$$

with $a_i \in \mathbb{C}$. Therefore, we follow a different approach to describe the Cartan subalgebra (see e.g. [40]). Because there is a difference in behaviour between even-dimensional and odd-dimensional orthogonal Lie algebras, we discuss these two cases separately.

Even-dimensional case

In order to obtain a Cartan subalgebra that consists of diagonal matrices, consider the nondegenerate bilinear form

$$Q(x,y) = x^T M y$$

with $x, y \in \mathbb{C}^{n\times 1}$ and

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The Lie group is $SO(2n,\mathbb{R}) = \{A \in GL(2n,\mathbb{R}) \,|\, A^T M A = M,\ \det A = 1\}$, which implies that $\mathfrak{so}(2n,\mathbb{C}) = \{X \in \mathfrak{gl}(2n,\mathbb{C}) \,|\, X^T M + MX = 0\}$.

For every $X = (x_{i,j}) \in \mathfrak{so}(2n, \mathbb{C})$, we have in particular that $x_{i,i} + x_{n+i,n+i} = 0$ for all $1 \leq i \leq n$. This means that automatically trace $X = 0$. In general, this Lie algebra consists of matrices of the form

$$\mathfrak{so}(2n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mid B^T = -B,\ C^T = -C \right\}$$

with A, B and C complex $n \times n$-matrices. Basis elements for $\mathfrak{so}(2n, \mathbb{C})$ are

$$\begin{aligned} X_{ij} &= E_{ij} - E_{n+j,n+i}, & 1 \leq i, j \leq n \\ Y_{ij} &= E_{i,n+j} - E_{j,n+i}, & 1 \leq j < i \leq n \\ Z_{ij} &= E_{n+i,j} - E_{n+j,i}, & 1 \leq j < i \leq n. \end{aligned} \tag{2.27}$$

The Cartan subalgebra $\mathfrak{h}$ is the n-dimensional space of diagonal matrices X_{ii}, which we normalise as follows:

$$\mathfrak{h} = \left\{ \beta_i \in \mathfrak{so}(2n, \mathbb{C}) \mid \beta_i = \frac{1}{\sqrt{2}}(E_{ii} - E_{n+i,n+i}),\ 1 \leq i \leq n \right\}.$$

Using the inner product (2.22), we have $\langle \beta_i, \beta_i \rangle = 1$, $\langle \beta_i, \beta_j \rangle = 1$ for $i \neq j$. The action of $\mathfrak{h}$ on the basis elements is given by

$$\begin{array}{lll} [\beta_i, X_{ij}] = X_{ij} & [\beta_i, Y_{ij}] = Y_{ij} & [\beta_i, Z_{ij}] = -Z_{ij} \\ [\beta_j, X_{ij}] = -X_{ij} & [\beta_j, Y_{ij}] = Y_{ij} & [\beta_j, Z_{ij}] = -Z_{ij} \\ [\beta_k, X_{ij}] = 0 & [\beta_k, Y_{ij}] = 0 & [\beta_k, Z_{ij}] = 0 \end{array} \tag{2.28}$$

for $k \neq i, j$. The weights of a representation of $\mathfrak{so}(2n, \mathbb{C})$ can be found through the following procedure:

1. determine the roots α
2. determine the co-roots H_α
3. calculate the *weight lattice*, which is the set of linear functionals that are integer-valued on all H_α and contains all weights of all representations.

First, we apply definition (2.26) to find the roots $\alpha \in \mathfrak{h}$ that correspond to X_{ij}. By means of (2.28) and $[\beta_i, X_{ij}] = \alpha(\beta_i) X_{ij} = X_{ij}$, we have $\alpha(\beta_i) = 1$. Similarly, we have $\alpha(\beta_j) = -1$, which leads to the roots

$$\alpha = \beta_i - \beta_j.$$

In the same way, the respective roots for Y_{ij} and Z_{ij} are given by

$$\alpha = \beta_i + \beta_j$$

and

$$\alpha = -\beta_i - \beta_j.$$

Next, we calculate the co-roots H_α in $\mathfrak{h}$. These can be found by means of the properties in Fact 4. Since

$$\begin{array}{ll} [X_{ij}, X_{ji}] = \beta_i - \beta_j & [Z_{ij}, Y_{ji}] = \beta_i + \beta_j \\ [\beta_i - \beta_j, X_{ij}] = 2X_{ij} & [\beta_i + \beta_j, Z_{ij}] = 2Z_{ij} \\ [\beta_i - \beta_j, X_{ji}] = -2X_{ji} & [\beta_i + \beta_j, Y_{ij}] = -2Y_{ij}, \end{array}$$

we have

$$H_{\beta_i - \beta_j} = \beta_i - \beta_j, \quad \mathfrak{g}_{\beta_i - \beta_j} = \{aX_{ij} \,|\, a \in \mathbb{C}\}, \quad \mathfrak{g}_{-\beta_i + \beta_j} = \{aX_{ji} \,|\, a \in \mathbb{C}\}$$

and

$$H_{\beta_i + \beta_j} = \beta_i + \beta_j, \quad \mathfrak{g}_{\beta_i + \beta_j} = \{aY_{ij} \,|\, a \in \mathbb{C}\}, \quad \mathfrak{g}_{-\beta_i - \beta_j} = \{aZ_{ij} \,|\, a \in \mathbb{C}\}.$$

Finally, we determine the weight lattice. As every weight of a representation of $\mathfrak{so}(2n, \mathbb{C})$ is an integral element, we have to look for all $\beta \in \mathfrak{h}$ such that $\langle \beta, H_\alpha \rangle$ is an integer for each root α. The weight lattice is generated by the β_k $(1 \le k \le n)$ together with the element $\frac{1}{2}(\beta_1 + \cdots + \beta_n)$. Indeed,

$$\begin{aligned} &\langle \beta_i + \beta_j, \beta_k \rangle = \delta_{ik} + \delta_{jk} \\ &\langle \beta_i + \beta_j, \frac{1}{2}(\beta_1 + \cdots + \beta_n) \rangle = \frac{1}{2}\langle \beta_i, \beta_i \rangle + \frac{1}{2}\langle \beta_j, \beta_j \rangle = 1 \\ &\langle \beta_i - \beta_j, \beta_k \rangle = \delta_{ik} - \delta_{jk} \\ &\langle \beta_i - \beta_j, \frac{1}{2}(\beta_1 + \cdots + \beta_n) \rangle = \frac{1}{2}\langle \beta_i, \beta_i \rangle - \frac{1}{2}\langle \beta_j, \beta_j \rangle = 0. \end{aligned}$$

Hence there are two types of weights:

- the integer weights $k_1\beta_1 + \cdots + k_n\beta_n$ with k_i integers, denoted in short by

$$(k_1, \ldots, k_n)$$

- the half-integer weights $\frac{1}{2}(l_1\beta_1 + \cdots + l_n\beta_n)$ with l_i odd, denoted by

$$(\frac{l_1}{2}, \ldots, \frac{l_n}{2}).$$

Some examples of representations of (half-)integer weight in Clifford analysis are discussed in section 2.3.6.

The positive roots of $\mathfrak{so}(2n,\mathbb{C})$, relative to $\mathfrak{h}$, are the matrices $\beta_i - \beta_j$ and $\beta_i + \beta_j$ for $1 \leq i < j \leq n$. The positive simple roots are given by

$$\beta_1 - \beta_2,\ \beta_2 - \beta_3,\ \ldots,\ \beta_{n-1} - \beta_n,\ \beta_{n-1} + \beta_n. \tag{2.29}$$

The closed fundamental Weyl chamber is the set $\langle \beta, H_\alpha \rangle \geq 0$ for all positive simple roots α, i.e.

$$\{a_1\beta_1 + \cdots + a_n\beta_n \mid a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|\}. \tag{2.30}$$

To end this section, we determine the fundamental weights. Algebraically, these are given by $\mu_i \in \mathfrak{h}$ $(1 \leq i \leq n)$ such that $\langle \mu_i, H_{\alpha_j} \rangle = \delta_{ij}$ with α_j the positive simple roots ordered as in (2.29). Denoting the weights μ_i in general by $\mu_i = a_1\beta_1 + \cdots + a_n\beta_n$, we fix μ_1 as follows:

$$\langle \mu_1, \beta_1 - \beta_2 \rangle = 1,\ \langle \mu_1, \beta_2 - \beta_3 \rangle = 0,\ \ldots\ ,\ \langle \mu_1, \beta_{n-1} + \beta_n \rangle = 0,$$

which leads to $a_1 = 1, a_2 = \ldots = a_n = 0$. In the same way, define μ_2 as

$$\langle \mu_2, \beta_1 - \beta_2 \rangle = 0,\ \langle \mu_2, \beta_2 - \beta_3 \rangle = 1,\ \ldots\ ,\ \langle \mu_2, \beta_{n-1} + \beta_n \rangle = 0,$$

which leads to $a_1 = a_2 = 1, a_3 = \ldots = a_n = 0$. This continues until we reach μ_n, which is defined as

$$\langle \mu_n, \beta_1 - \beta_2 \rangle = 0,\ \langle \mu_n, \beta_2 - \beta_3 \rangle = 0,\ \ldots\ ,\ \langle \mu_n, \beta_{n-1} + \beta_n \rangle = 1;$$

this leads to $a_1 = \ldots = a_n = \frac{1}{2}$. Summarising, the fundamental weights are

$$\begin{aligned}
\mu_1 &= (1,0,0,\ldots,0)\\
\mu_2 &= (1,1,0,\ldots,0)\\
&\ \vdots\\
\mu_{n-2} &= (1,1,\ldots,1,0,0)\\
\mu_{n-1} &= (\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})\\
\mu_n &= (\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2}).
\end{aligned}$$

Every weight in the closed fundamental Weyl chamber can be written as a linear combination of the fundamental weights μ_i with non-negative integer coefficients. Because the fundamental weights are the weights met along the edges of the Weyl chamber, there is an alternative way to find them. The Weyl chamber of the orthogonal Lie algebra $\mathfrak{so}(2n, \mathbb{C})$ is a simplicial cone with faces the n planes given by $a_1 = a_2, \ldots, a_{n-1} = a_n$ and $a_{n-1} = -a_n$. The edges of the Weyl chamber are $\beta_1 + \cdots + \beta_i$ (for all $1 \leq i \leq n-2$), $\beta_1 + \cdots + \beta_{n-1} - \beta_n$ and $\beta_1 + \cdots + \beta_{n-1} + \beta_n$.

Odd-dimensional case

Because the odd-dimensional case is very similar to the even-dimensional case, we merely point out the differences. We now work with the $(2n+1) \times (2n+1)$-matrix

$$M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the Lie algebra consists of matrices of the form

$$\mathfrak{so}(2n+1, \mathbb{C}) = \Big\{ \begin{pmatrix} A & B & a \\ C & -A^T & b \\ -a^T & -b^T & 0 \end{pmatrix} \mid B^T = -B,\ C^T = -C \Big\}$$

with a, b in $\mathbb{C}^{n\times 1}$. A basis for $\mathfrak{so}(2n+1, \mathbb{C})$ is given by the matrices X_{ij}, Y_{ij}, Z_{ij} from (2.27) together with the basis elements

$$\begin{aligned} V_i &= -E_{i,2n+1} + E_{2n+1,n+i} \\ U_i &= -E_{n+i,2n+1} + E_{2n+1,i} \end{aligned}$$

for $1 \leq i \leq n$. Since

$$\begin{array}{ll} [\beta_i, U_i] = U_i & [\beta_i, V_i] = -V_i \\ [\beta_k, U_i] = 0 & [\beta_k, V_i] = 0 \end{array} \tag{2.31}$$

for $k \neq i$, the roots corresponding to U_i and V_i are $\alpha = \beta_i$ and $\alpha = -\beta_i$, respectively. The corresponding co-roots can be found by considering

$$[U_i, V_i] = \beta_i,\ [\beta_i, U_i] = U_i,\ [\beta_i, V_i] = -V_i.$$

Because H_α must satisfy $\alpha(H_\alpha) = 2$, we have

$$H_{\beta_i} = 2\beta_i, \quad \mathfrak{g}_{\beta_i} = \{aU_i \,|\, a \in \mathbb{C}\}, \quad \mathfrak{g}_{-\beta_i} = \{aV_i \,|\, a \in \mathbb{C}\}.$$

In the odd-dimensional case, the weight lattice is again generated by the β_k $(1 \le k \le n)$ together with the element $\frac{1}{2}(\beta_1 + \cdots + \beta_n)$. The positive roots are the matrices $\beta_i - \beta_j$ and $\beta_i + \beta_j$ and β_i $(1 \le i < j \le n)$. The positive simple roots are given by

$$\beta_1 - \beta_2,\ \beta_2 - \beta_3,\ \ldots,\ \beta_{n-1} - \beta_n,\ \beta_n;$$

the closed fundamental Weyl chamber is the set

$$\{a_1\beta_1 + \cdots + a_n\beta_n \,|\, a_1 \ge a_2 \ge \cdots \ge a_n \ge 0\} \tag{2.32}$$

and the fundamental weights are

$$\begin{aligned} \mu_1 &= (1, 0, 0, \ldots, 0) \\ \mu_2 &= (1, 1, 0, \ldots, 0) \\ &\vdots \\ \mu_{n-1} &= (1, 1, \ldots, 1, 1, 0) \\ \mu_n &= (\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}). \end{aligned}$$

Remark 10. *Recall that $\mathfrak{so}(m, \mathbb{C}) \cong \mathbb{C}_m^{(2)}$. Hence, we can write (2.28) and (2.31) in the language of Clifford algebras by identifying*

$$\beta_i = \frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i \in \mathfrak{h} \tag{2.33}$$

and

$$X_{ij} = \mathfrak{f}_j^\dagger \mathfrak{f}_i,\quad Y_{ij} = \mathfrak{f}_i \mathfrak{f}_j,\quad Z_{ij} = \mathfrak{f}_i^\dagger \mathfrak{f}_j^\dagger,\quad U_i = \mathfrak{f}_i e_m,\quad V_i = \mathfrak{f}_i^\dagger e_m.$$

Using the information of this remark, we give an example of $\mathfrak{so}(m, \mathbb{C})$-representations that correspond to the integer fundamental weights above. It follows from (2.5) and (2.18) that the derived h-action of the Spin group on $x \in \mathbb{C}_m^{(k)}$ is given by $[h, x]$. Since

$$\begin{aligned} \big[\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i, \mathfrak{f}_{j_1}\big] &= \delta_{ij_1} \mathfrak{f}_i \\ \big[\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i, \mathfrak{f}_{j_1} \ldots \mathfrak{f}_{j_k}\big] &= \delta_{i,j_1} \mathfrak{f}_i \mathfrak{f}_{j_2} \ldots \mathfrak{f}_{j_k} + \cdots + \delta_{i,j_k} \mathfrak{f}_{j_1} \ldots \mathfrak{f}_{j_{k-1}} \mathfrak{f}_i, \end{aligned}$$

a highest weight vector for $h : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{C}_m^{(k)})$ is given by $\mathfrak{f}_1 \ldots \mathfrak{f}_k$ with highest weight $\mu_k = (1, \ldots, 1, 0, \ldots, 0)$. Examples of $\mathfrak{so}(m, \mathbb{C})$-representations that correspond to the half-integer fundamental weights are discussed in the next section.

2.3.6 Irreducible representations of Spin(m) in Clifford analysis

Spinor space

Recall the two representations of the spinor space in the even-dimensional case $m = 2n$:

$$\begin{aligned}\mathbb{S}_{2n}^{+} &= \mathrm{Alg}_{\mathbb{C}}^{+}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)I \\ \mathbb{S}_{2n}^{-} &= \mathrm{Alg}_{\mathbb{C}}^{-}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)I.\end{aligned}$$

We prove that $\mathbb{S}_{2n}^{\pm}$ are the irreducible representations of the Spin group or, equivalently, its complex Lie algebra $\mathfrak{so}(2n, \mathbb{C})$, with highest weights $\left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right)$, respectively.

The Cartan algebra is given by

$$\mathfrak{h} = \left\{\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i \,|\, 1 \leq i \leq n\right\}. \tag{2.34}$$

Let I be the primitive idempotent (2.3). The basis elements of $\mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)I$ are of the form

$$v = \mathfrak{f}_{j_1}^\dagger \ldots \mathfrak{f}_{j_k}^\dagger I \tag{2.35}$$

with $1 \leq j_1 < \ldots < j_k \leq n$. Denote by J the set $\{j_1, \ldots, j_k\}$. If $J = \emptyset$, we have $v = I$. The Cartan subalgebra acts on v by left multiplication. Since

$$\mathfrak{f}_i^\dagger \mathfrak{f}_i v \;=\; [\mathfrak{f}_i^\dagger \mathfrak{f}_i, v] \;=\; \delta_{i,j_1} \mathfrak{f}_i^\dagger \mathfrak{f}_{j_2}^\dagger \ldots \mathfrak{f}_{j_k}^\dagger I + \cdots + \delta_{i,j_k} \mathfrak{f}_{j_1}^\dagger \ldots \mathfrak{f}_{j_{k-1}}^\dagger \mathfrak{f}_i^\dagger I,$$

the elements (2.35) are eigenvectors for $\mathfrak{f}_i^\dagger \mathfrak{f}_i$ $(1 \leq i \leq n)$ with integral eigenvalues, which makes them weight vectors. We have

$$\left(\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i\right) v \;=\; \begin{cases} -\frac{1}{2} v & \text{if } i \in J \\ +\frac{1}{2} v & \text{if } i \notin J. \end{cases}$$

Therefore, v spans a weight space with weight

$$\frac{1}{2}\left(-\sum_{i \in J} \beta_i + \sum_{i \notin J} \beta_i\right). \tag{2.36}$$

In particular, the eigenvalue that corresponds to $v = I$ is

$$(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$$

and the eigenvalue corresponding to $v = \mathfrak{f}_n^\dagger I$ is

$$(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}).$$

These are the highest weight vectors and highest weights for the spinor spaces $\mathbb{S}_{2n}^+ = \mathrm{Alg}_{\mathbb{C}}^+(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)I$ and $\mathbb{S}_{2n}^- = \mathrm{Alg}_{\mathbb{C}}^-(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)I$, respectively.

In the same way, $\mathbb{S} = \mathrm{Alg}_{\mathbb{C}}(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)I$, with I as in (2.4), is the irreducible representation of $\mathfrak{so}(2n+1, \mathbb{C})$ with highest weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\right)$, the action being left multiplication.

Spherical harmonics and spherical monogenics

A mentioned before, we put $m = 2n + 1$ odd, for convenience. It is well-known (see e.g. [42, 73]) that the vector space $\mathcal{M}_k$ of spherical monogenics of degree k forms an irreducible representation of $\mathrm{Spin}(m)$. In this section, we will show that $\mathcal{M}_k$ corresponds to an irreducible representation of the L-representation with highest weight $(k+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. This highest weight is calculated by means of the derived action dL of $\mathfrak{so}(m, \mathbb{C})$.

Since $[L(s), \partial_x] = 0$, the vector space $\mathcal{M}_k$ is invariant with respect to the Spin group. Due to the irreducibility of this vector space, $\mathcal{M}_k$ is determined by specifying its highest weight vector

$$v = \langle x, \mathfrak{f}_1 \rangle^k I$$

with I the primitive idempotent from (2.4). It is easy to see that $v \in \mathcal{M}_k$. The Cartan algebra $\mathfrak{h}$ is the same as in (2.34).

Using the derived representation (2.17), the action of an element of the Cartan subalgebra is

$$dL(\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i) = (\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i) + 2\langle x, \mathfrak{f}_i^\dagger \rangle \langle \partial_x, \mathfrak{f}_i \rangle - 2\langle x, \mathfrak{f}_i \rangle \langle \partial_x, \mathfrak{f}_i^\dagger \rangle$$

from which

$$dL(\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i) \langle x, \mathfrak{f}_1 \rangle^k I \; = \; \frac{1}{2} \langle x, \mathfrak{f}_1 \rangle^k I + k \langle x, \mathfrak{f}_i \rangle \langle x, \mathfrak{f}_1 \rangle^{k-1} \delta_{ik} I.$$

If we make a distinction between $i = 1$ and $i > 1$

$$\begin{aligned} dL(\frac{1}{2} - \mathfrak{f}_1^\dagger \mathfrak{f}_1)\langle x, \mathfrak{f}_1\rangle^k I &= (k + \frac{1}{2})\langle x, \mathfrak{f}_1\rangle^k I \\ dL(\frac{1}{2} - \mathfrak{f}_i^\dagger \mathfrak{f}_i)\langle x, \mathfrak{f}_1\rangle^k I &= \frac{1}{2}\langle x, \mathfrak{f}_1\rangle^k I, \end{aligned}$$

we find the highest weight $(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

Similarly, the vector space $\mathcal{H}_k$ of spherical harmonics of degree k forms an irreducible representation of the Spin group (resp. $\mathfrak{so}(2n + 1, \mathbb{C})$) with highest weight $(k, 0, \ldots, 0)$ of the H-representation (resp. dH-representation).

Remark 11. *Rather than referring to the derived representation of its complex Lie algebra $\mathfrak{so}(m, \mathbb{C})$, we will usually talk about highest weights of representations of Spin(m).*

Simplicial harmonics and simplicial monogenics

It was shown in [73, 42] that irreducible finite-dimensional representations of the Spin group of integer (resp. half-integer) weight can be realised in terms of harmonic (resp. monogenic) polynomials in several vector variables in Clifford analysis. We refer to these representations as irreducible Spin(m)-modules.

Let N be an integer satisfying $1 \leq N \leq \lfloor m/2 \rfloor$. The Dirac operator with respect to the Clifford vectors u_i is denoted ∂_i instead of ∂_{u_i} $(i = 1, \ldots, N)$.

Definition 1. *A function $f : \mathbb{R}^{Nm} \to \mathbb{C}$, $(u_1, \ldots, u_N) \mapsto f(u_1, \ldots, u_N)$ is simplicial harmonic if the following conditions are satisfied:*

$$\begin{aligned} \langle \partial_i, \partial_j \rangle f &= 0, \qquad i, j = 1, \ldots, N \\ \langle u_i, \partial_j \rangle f &= 0, \qquad 1 \leq i < j \leq N. \end{aligned}$$

The vector space of simplicial harmonic polynomials, k_i-homogeneous in the variable u_i is denoted by $\mathcal{H}_{k_1,\ldots,k_N}$ (with integers $k_1 \geq \ldots \geq k_N \geq 0$).

Definition 2. *A function $f : \mathbb{R}^{Nm} \to \mathbb{S}$, $(u_1, \ldots, u_N) \mapsto f(u_1, \ldots, u_N)$ is simplicial monogenic if the following conditions are satisfied:*

$$\begin{aligned} \partial_i f &= 0, \qquad i = 1, \ldots, N \\ \langle u_i, \partial_j \rangle f &= 0, \qquad 1 \leq i < j \leq N. \end{aligned}$$

The vector space of simplicial monogenic polynomials, k_i-homogeneous in the variable u_i is denoted $\mathcal{S}_{k_1,\ldots,k_N}$ (with integers $k_1 \geq \ldots \geq k_N \geq 0$).

Clearly, if a function is simplicial monogenic in an open region Ω of $\mathbb{R}^{Nm}$, then each of its scalar components is simplicial harmonic in Ω; in other words, $\mathcal{S}_{k_1,\ldots,k_N} \subset \mathcal{H}_{k_1,\ldots,k_N} \otimes \mathbb{S}$.

The vector space $\mathcal{H}_{k_1,\ldots,k_N}$ is an irreducible representation of $\mathrm{Spin}(m)$. The highest weight of the H-representation acting on $f \in \mathcal{H}_{k_1,\ldots,k_N}$, i.e.

$$H(s)f(u_1,\ldots,u_N) = f(\overline{s}u_1s,\ldots,\overline{s}u_Ns)$$

is given by $(k_1,\ldots,k_N,0,\ldots,0)$. The highest weight of the irreducible $\mathrm{Spin}(m)$-module $\mathcal{S}_{k_1,\ldots,k_N}$ is given by $(k_1+\frac{1}{2},\ldots,k_N+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ with respect to the L-representation on $f \in \mathcal{S}_{k_1,\ldots,k_N}$:

$$L(s)f(u_1,\ldots,u_N) = sf(\overline{s}u_1s,\ldots,\overline{s}u_Ns).$$

Taking $N=1$ in these definitions, we obtain the spherical harmonics $\mathcal{H}_{k_1}$ of degree k_1 and the spherical monogenics $\mathcal{M}_{k_1}$ of degree k_1.

Remark 12. *In case $N = \frac{m}{2}$ the spaces $\mathcal{H}_{k_1,\ldots,k_N}$ and $\mathcal{S}_{k_1,\ldots,k_N}$ are irreducible Pin(m)-modules; under the action of Spin(m) they split in two irreducible summands ([73]).*

The second condition in Definition 1 (resp. 2) implies that an arbitrary polynomial $p \in \mathcal{H}_{k_1,\ldots,k_N}$ (resp. $\mathcal{S}_{k_1,\ldots,k_N}$) can be identified with a $\mathbb{C}$-valued (resp. $\mathbb{S}$-valued) polynomial f depending only of a number of specific wedge products of the vector variables:

$$p(u_1,u_2,\ldots,u_N) = f(u_1, u_1\wedge u_2, u_1\wedge u_2\wedge u_3,\ldots,u_1\wedge u_2\wedge\ldots\wedge u_N).$$

The highest weight vector of $\mathcal{S}_{k_1,\ldots,k_N}$ is given by

$$\langle u_1,\mathfrak{f}_1\rangle^{k_1-k_2}\langle u_1\wedge u_2,\mathfrak{f}_1\wedge\mathfrak{f}_2\rangle^{k_2-k_3}\ldots\langle u_1\wedge\cdots\wedge u_N,\mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_N\rangle^{k_N} I \quad (2.37)$$

with, for all $1 \leq j \leq N$,

$$\begin{aligned}\langle u_1\wedge\ldots\wedge u_j,\mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_j\rangle &:= \det\begin{pmatrix}\langle u_1,\mathfrak{f}_1\rangle & \ldots & \langle u_1,\mathfrak{f}_j\rangle\\ \vdots & \ddots & \vdots\\ \langle u_j,\mathfrak{f}_1\rangle & \ldots & \langle u_j,\mathfrak{f}_j\rangle\end{pmatrix}\\ &= \sum_{\sigma\in\mathfrak{S}_j}\mathrm{sgn}(\sigma)\langle u_{\sigma(1)},\mathfrak{f}_1\rangle\cdots\langle u_{\sigma(j)},\mathfrak{f}_j\rangle.\end{aligned}$$

The highest weight vector of of $\mathcal{H}_{k_1,\ldots,k_N}$ is obtained by omitting the primitive idempotent I in (2.37).

Remark 13 (Notation)**.** *In what follows, $\mathcal{P}_{k_1,\ldots,k_N}(\mathbb{R}^{Nm}, V)$ denotes the space of V-valued polynomials in N variables of homogeneity degree k_i ($k_i \in \mathbb{N}$) in the ith variable.*

Remark 14. *As opposed to the one-variable case, the conditions involving the operators $\langle u_i, \partial_j \rangle$ in the definition of simplicial monogenic polynomials are needed in order to obtain an irreducible module for Spin(m). For example, $\mathcal{M}_k$ is an irreducible module, while*

$$\mathcal{M}_{k_1,k_2} := \{f \in \mathcal{P}_{k_1,k_2}(\mathbb{R}^{2m}, \mathbb{S}) \mid \partial_1 f = \partial_2 f = 0\}$$

can be decomposed into irreducible Spin(m)-modules as

$$\mathcal{M}_{k_1,k_2} = \bigoplus_{j=0}^{k_1-k_2} \langle u_2, \partial_1 \rangle^j \mathcal{S}_{k_1+j,k_2-j}.$$

For more details, see section 2.4.1 and [18].

Remark 15 (Short notation)**.** *We will often need to refer to the highest weight of a representation of the Spin group; to that end we introduce the short notation $(k_1, \ldots, k_N)$ for $(k_1, \ldots, k_N, 0, \ldots, 0)$ and we denote by $(k_1, \ldots, k_N)'$ the half-integer highest weight $(k_1 + \frac{1}{2}, \ldots, k_N + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Because we usually deal with the H- or L-representation, we simply say 'highest weight' without explicitly mentioning the action of the Spin group.*

2.3.7 Useful theorems

For the proofs of the following theorems, we refer to e.g. [40, 49]. For each weight, let $P(\mu)$ be the number of ways to write μ as a sum of positive roots and put $P(0) = 1$. Let V be a finite-dimensional representation and $\mathbb{V}_\lambda$ an irreducible representation with highest weight λ. The multiplicity of $\mathbb{V}_\lambda$ in V is denoted $n_\lambda(V)$ and the multiplicity of a weight μ in $\mathbb{V}_\lambda$ is denoted $m_\mu(\lambda)$.

Theorem 6 (Kostant)**.** *Suppose that $\mathbb{V}_{\mu_0}$ is a finite-dimensional irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}$ with highest weight μ_0. If μ is a weight of $\mathbb{V}_{\mu_0}$, then the multiplicity $m_{\mu_0}(\mu)$ of this weight in $\mathbb{V}_{\mu_0}$ is given by*

$$m_{\mu_0}(\mu) = \sum_{w \in W} sgn(w) P\big(w \cdot (\mu_0 + \delta) - (\mu + \delta)\big)$$

where δ is half the sum of the positive roots.

For the proof of the next proposition, we refer to [21].

Proposition 5. *If ν is a dominant integral weight such that $n_\nu(\mathbb{V}_\lambda \otimes \mathbb{V}_\mu) > 0$, then there is a weight $\tilde{\mu}$ of $\mathbb{V}_\mu$ such that $\nu = \lambda + \tilde{\mu}$. If this is the case, we have $n_\nu(\mathbb{V}_\lambda \otimes \mathbb{V}_\mu) \leq m_{\lambda-\nu}(\mu)$.*

Theorem 7 (Klimyk's formula)**.** *The multiplicity of an irreducible representation with weight ν in the tensor product $\mathbb{V}_\lambda \otimes \mathbb{V}_\mu$ is given by*

$$n_\nu(\mathbb{V}_\lambda \otimes \mathbb{V}_\mu) = \sum_{w \in W} sgn(w) m_{\nu+\delta-w(\lambda+\delta)}(\mu)$$

where δ is half the sum of the positive roots.

Theorem 8 (Weyl dimension formula)**.** *Suppose that π is an irreducible representation of $\mathfrak{g}$ with highest weight μ. The dimension of π is given by*

$$\dim \pi = \frac{\prod_{\alpha \in R^+} \langle \alpha, \mu + \delta \rangle}{\prod_{\alpha \in R^+} \langle \alpha, \delta \rangle}$$

where R^+ denotes the set of positive roots and δ is half the sum of the positive roots.

As an example, the dimension of an irreducible representation π of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight (m_1, m_2) is given by

$$\dim \pi = \frac{1}{2}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2).$$

If $(m_1, m_2) = (1, 1)$ we find that $\dim \pi = 8$, which is in agreement with the results in section 2.3.3.

Dimension of $\mathcal{M}_k$

To illustrate the Weyl dimension formula, we calculate the dimension of $\mathcal{M}_k$ as a vector space over $\mathbb{C}$:

$$\dim \mathcal{M}_k = \frac{\prod_{\alpha \in R^+} \langle \alpha, \mu + \delta \rangle}{\prod_{\alpha \in R^+} \langle \alpha, \delta \rangle}. \tag{2.38}$$

In this case, $\delta = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2})$ with its jth entry given by $n - j + \frac{1}{2}$. The highest weight is $\mu = (k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$ and $\mu + \delta = (n+k, n-1, n-2, \ldots, 2, 1)$.

First, we calculate the contribution to (2.38) of the roots $\beta_1-\beta_j$ $(2 \le j \le n)$. We have

$$\langle \beta_1 - \beta_j, \mu+\delta\rangle = k+j-1, \qquad \langle \beta_1 - \beta_j, \delta\rangle = j-1,$$

leading to

$$\prod_{j=2}^{n} \frac{\langle \beta_1 - \beta_j, \mu+\delta\rangle}{\langle \beta_1 - \beta_j, \delta\rangle} = \frac{(k+n-1)!}{k!(n-1)!}.$$

In a similar way, the roots $\beta_1+\beta_j$ $(2 \le j \le n)$ satisfy

$$\langle \beta_1 + \beta_j, \mu+\delta\rangle = 2n+k-j+1, \qquad \langle \beta_1 + \beta_j, \delta\rangle = 2n-j$$

which leads to

$$\prod_{j=2}^{n} \frac{\langle \beta_1 + \beta_j, \mu+\delta\rangle}{\langle \beta_1 + \beta_j, \delta\rangle} = \frac{(2n+k-1)!(n-1)!}{(n+k)!(2n-2)!}$$

and the contribution of the root β_1 is

$$\frac{\langle \beta_1, \mu+\delta\rangle}{\langle \beta_1, \delta\rangle} = \frac{2(n+k)}{2n-1}.$$

Since

$$\langle \beta_i - \beta_j, \mu+\delta\rangle = j-i = \langle \beta_i - \beta_j, \delta\rangle,$$

there is no contribution of the roots $\beta_i-\beta_j$ $(1<i<j<n)$. The roots $\beta_i+\beta_j$ $(1<i<j<n)$ satisfy

$$\langle \beta_i + \beta_j, \mu+\delta\rangle = 2n+2-i-j, \qquad \langle \beta_i + \beta_j, \delta\rangle = 2n+1-i-j,$$

from which

$$\prod_{1<i<j}^{n} \frac{\langle \beta_i + \beta_j, \mu+\delta\rangle}{\langle \beta_i + \beta_j, \delta\rangle} = \frac{(2n-3)(2n-5)\cdots 5\cdot 3}{(n-1)!}.$$

Finally, the roots β_i $(2 \le i \le n)$ satisfy

$$\langle \beta_i, \mu+\delta\rangle = n-i+1, \qquad \langle \beta_i, \delta\rangle = \frac{1}{2}(2n-2j+1)$$

and add the following contribution to (2.38):

$$\prod_{i=2}^{n} \frac{\langle \beta_i, \mu+\delta\rangle}{\langle \beta_i + \beta_j, \delta\rangle} = 2^{n-1}\frac{(n-1)!}{(2n-3)(2n-5)\cdots 5\cdot 3}.$$

Gathering all these results, we find

$$\dim \mathcal{M}_k = 2^n \begin{pmatrix} k+2n-1 \\ k \end{pmatrix} = 2^n \begin{pmatrix} k+m-2 \\ k \end{pmatrix} \tag{2.39}$$

which can also be calculated by means of the CK-extension, see (3.9). Note that 2^n corresponds to the dimension of $\mathbb{S}$.

Dimension of $\mathcal{S}_{k,l}$ and $\mathcal{S}_{h,k,l}$

Calculating the dimension of the irreducible Spin(m)-module $\mathcal{S}_{k,l}$ is completely similar to the case of $\mathcal{M}_k$ but more involved. In this case, the highest weight is $(k+\frac{1}{2}, l+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, which leads to

$$\dim \mathcal{S}_{k,l} = 2^n \begin{pmatrix} k+2n-2 \\ k+1 \end{pmatrix} \begin{pmatrix} l+2n-3 \\ l \end{pmatrix} \frac{(2n+k+l-1)(k-l+1)}{(2n-1)(2n-2)}. \tag{2.40}$$

In case of $\mathcal{S}_{h,k,l}$, which is an irreducible Spin(m)-module with highest weight $(h+\frac{1}{2}, k+\frac{1}{2}, l+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, we find

$$\begin{aligned} \dim \mathcal{S}_{h,k,l} = 2^n & \begin{pmatrix} h+2n-3 \\ h+2 \end{pmatrix} \begin{pmatrix} k+2n-4 \\ k+1 \end{pmatrix} \begin{pmatrix} l+2n-5 \\ l \end{pmatrix} \\ & \cdot (2n+h+k-1)(2n+h+l-2)(2n+k+l-3) \\ & \cdot (h-k+1)(h-l+2)(k-l+1) \frac{(2n-5)!}{(2n-1)!} \frac{(2n-5)!}{(2n-3)!}. \end{aligned} \tag{2.41}$$

These results will be used in the next chapters.

Dimension of $\mathcal{S}_{\lambda_k}$ and $\mathcal{S}_{\lambda_{h,k}}$

As a final example of the Weyl dimension formula, we present another result, which will come in handy in chapter 5. Let h be an integer with $h \geq 1$. We adopt the following notations:

$$\lambda_k := (\underbrace{\frac{3}{2}, \cdots, \frac{3}{2}}_{k}, \frac{1}{2}, \cdots, \frac{1}{2}) = (\underbrace{1, \cdots, 1}_{k})' \tag{2.42}$$

$$\lambda_{h,k} := (h+\frac{1}{2}, \underbrace{\frac{3}{2}, \cdots, \frac{3}{2}}_{k}, \frac{1}{2}, \cdots, \frac{1}{2}) = (h, \underbrace{1, \cdots, 1}_{k})'. \tag{2.43}$$

If we denote the vector spaces of simplicial monogenics corresponding to these highest weights by $\mathcal{S}_{\lambda_k}$ and $\mathcal{S}_{\lambda_{h,k}}$, respectively, we have

$$\dim \mathcal{S}_{\lambda_k} = 2^n \binom{2n+1}{k} \frac{2n-2k+2}{2n-k+2} \tag{2.44}$$

$$\dim \mathcal{S}_{\lambda_{h,k}} = 2^n \binom{h+2n}{k} \binom{2n+h-k-1}{h-1} \frac{2n-2k}{h+k}. \tag{2.45}$$

2.4 Monogenics in several variables

2.4.1 Double monogenics

Let $k \geq l$ be positive integers. In Remark 14 the vector space of double monogenics in the variables (x, u) was introduced as

$$\mathcal{M}_{k,l} = \{f \in \mathcal{P}_{k,l}(\mathbb{R}^{2m}, \mathbb{S}) \,|\, \partial_x f = \partial_u f = 0\}.$$

From [18] we know its decomposition into spaces of simplicial monogenics:

$$\mathcal{M}_{k,l} = \bigoplus_{j=0}^{l} \langle u, \partial_x \rangle^j \mathcal{S}_{k+j,l-j}. \tag{2.46}$$

It is easy to verify that for all $0 \leq j \leq l$, we have

$$\langle u, \partial_x \rangle^j \mathcal{S}_{k+j,l-j} \hookrightarrow \mathcal{M}_{k,l}.$$

Recall that $\{\mathbb{E}_x - \mathbb{E}_u, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle\} \cong \mathfrak{sl}(2, \mathbb{C})$ and note that $\mathcal{S}_{k,l}$ is the highest weight vector of an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module of dimension $k - l + 1$.

2.4.2 Triple monogenics

Define the space of triple monogenics $\mathcal{M}_{h,k,l}$ in the variables x, u, v as

$$\mathcal{M}_{h,k,l} = \{f \in \mathcal{P}_{h,k,l}(\mathbb{R}^{3m}, \mathbb{S}) \,|\, \partial_x f = \partial_u f = \partial_v f = 0\}.$$

In this section, our aim is to find a decomposition of $\mathcal{M}_{h,k,l}$ that is similar to the decomposition (2.46) for $\mathcal{M}_{k,l}$. The Lie algebra of interest will now be $\mathfrak{sl}(3, \mathbb{C})$ and we consider the following representation in Clifford analysis:

$$\begin{array}{lll} H_1 = \mathbb{E}_x - \mathbb{E}_u & H_2 = \mathbb{E}_u - \mathbb{E}_v & \\ X_1 = \langle x, \partial_u \rangle & X_2 = \langle u, \partial_v \rangle & X_3 = [X_1, X_2] = \langle x, \partial_v \rangle \\ Y_1 = \langle u, \partial_x \rangle & Y_2 = \langle v, \partial_u \rangle & Y_3 = -[Y_1, Y_2] = \langle v, \partial_x \rangle. \end{array}$$

Note that

$$[Y_2, Y_1] = Y_3, \quad [Y_2, Y_3] = 0 = [Y_1, Y_3].$$

The weight of $\mathcal{S}_{p,q,r}$ is $(p-q, q-r)$, because

$$H_1\mathcal{S}_{p,q,r} = (p-q)\mathcal{S}_{p,q,r}, \quad H_2\mathcal{S}_{p,q,r} = (q-r)\mathcal{S}_{p,q,r}.$$

Furthermore, we have $X_1\mathcal{S}_{p,q,r} = X_2\mathcal{S}_{p,q,r} = 0$. By means of Proposition 3, we calculate the smallest invariant subspace $V \subset \mathcal{M}_{p,q,r}$ that contains $\mathcal{S}_{p,q,r}$ as a highest weight vector. This subspace can be found by acting with all possible combinations of Y_1 and Y_2 on $\mathcal{S}_{p,q,r}$. This is a finite-dimensional subspace, because $(Y_1)^{p-q+i}\mathcal{S}_{p,q,r} = 0$ and $(Y_2)^{q-r+i}\mathcal{S}_{p,q,r} = 0$ for $i > 0$. It is not difficult to see that V is an irreducible $\mathfrak{sl}(3,\mathbb{C})$-module with highest weight $(p-q, q-r)$.

The action of the other operators on a weight vector v with weight (a,b) is visualised in Figure 2.4. We have the following connection between the weight vectors and the corresponding weights for $i \in \mathbb{N}$:

$$\begin{array}{rcc} (Y_1)^i v & \to & (a-2i, b+i) \\ (Y_2)^i v & \to & (a+i, b-2i) \\ (Y_3)^i v & \to & (a-i, b-i) \\ (Y_1Y_3)^i v & \to & (a-3i, b) \\ (Y_2Y_3)^i v & \to & (a, b-3i) \end{array} \tag{2.47}$$

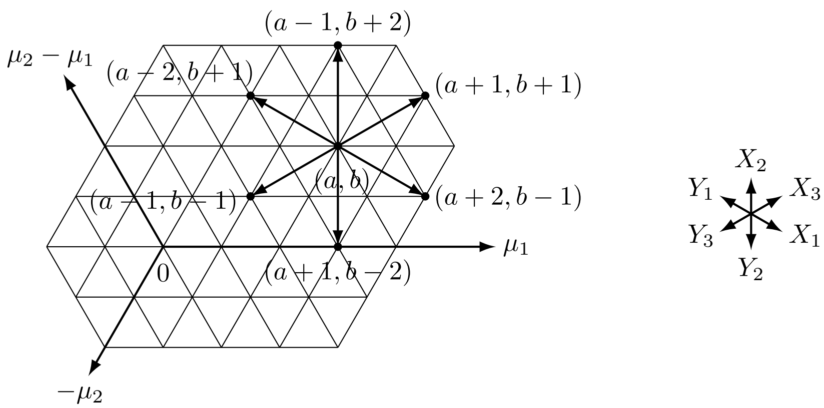


Figure 2.4: Action of $\mathfrak{sl}(3,\mathbb{C})$ on the weight (a,b).

As opposed to the case of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, not every weight in an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ has multiplicity 1. It was mentioned before that the weights occur in hexagon and triangle 'rings' with the following pattern: in the outermost ring the weights have multiplicity 1, the multiplicities then increase by 1 each time one moves one ring inwards in the direction of Y_3, until the rings become triangles and the multiplicities stabilise.

Similarly to (2.46) we are looking for the decomposition of the space of triple monogenics into simplicial monogenics. This means that we need to find all $\mathrm{Spin}(m)$-invariant mappings from spaces of simplicial monogenics into the space of triple monogenics. The theory on Howe-dual pairs (see [48]) states that the only candidates for these mappings are combinations of the basic invariants underlying Clifford analysis in three vector variables. These basic invariants are the operators that generate the Lie superalgebra $\mathfrak{osp}(1|6) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The even subalgebra $\mathfrak{g}_0 = \mathfrak{sp}(6)$ is spanned by the 21 elements

$$\begin{aligned}\mathfrak{g}_0 =& \big\{\mathbb{E}_x + \frac{m}{2}, \mathbb{E}_u + \frac{m}{2}, \mathbb{E}_v + \frac{m}{2}, X_1, X_2, X_3, Y_1, Y_2, Y_3\big\}\\ &\oplus \ \big\{|x|^2, |u|^2, |v|^2, \Delta_x, \Delta_u, \Delta_v\big\}\\ &\oplus \ \big\{\langle x,u\rangle, \langle x,v\rangle, \langle u,v\rangle, \langle \partial_x,\partial_u\rangle, \langle \partial_x,\partial_v\rangle, \langle \partial_u,\partial_v\rangle\big\}\end{aligned}$$

and the odd subspace equals

$$\mathfrak{g}_1 = \big\{x, u, v, \partial_x, \partial_u, \partial_v\big\}.$$

A crucial result in the general theory of Lie superalgebras states that any combination of these invariants, i.e. any element in the universal enveloping algebra $U(\mathfrak{osp}(1|6))$, can always be reordered (using the algebraic relations) so that we obtain (a sum of) elements of the form

$$(Z_1)^{a_1}(Z_2)^{a_2}\cdots(Z_{27})^{a_{27}} \in U\big(\mathfrak{osp}(1|6)\big),$$

with Z_j the generators for $\mathfrak{osp}(1|6)$ and $a_j \in \mathbb{N}\cup\{0\}$. It is shown in [10] that the only elements in the decomposition of $\mathcal{M}_{p,q,r}$ are of the form

$$Y\mathcal{S}_{h,k,l}$$

with Y a product of the operators Y_1, Y_2 and Y_3, in such a way that the degrees of homogeneity are correct.

Therefore, finding the decomposition for the triple monogenics is equivalent to answering the following questions:

1. Which vector spaces $\mathcal{S}_{h,k,l}$ with $h \geq k \geq l$ can be embedded in $\mathcal{M}_{p,q,r}$?
2. What are the multiplicities of the corresponding weights $(h-k, k-l)$?

An answer to these questions can be found in Lemmas 3 and 5.

Lemma 3. *The following spaces occur with multiplicity 1 in the decomposition of* $\mathcal{M}_{p,q,r}$*:*

- $\mathcal{S}_{p+i,q-i,r}$*, which is the highest weight vector of the unique irreducible representation of* $\mathfrak{sl}(3,\mathbb{C})$ *denoted* $\mathbb{U}_i$ *with highest weight* $(p-q+2i, q-r-i)$ *with* $i = 0, \ldots, q-r$
- $\mathcal{S}_{p,q+i,r-i}$*, which is the highest weight vector of the unique irreducible representation of* $\mathfrak{sl}(3,\mathbb{C})$ *denoted* $\mathbb{V}_i$ *with highest weight* $(p-q-i, q-r+2i)$ *with* $i = 1, \ldots, \min(r, p-q)$
- $\mathcal{S}_{p+q+r-2i,i,i}$*, which is the highest weight vector of the unique irreducible representation of* $\mathfrak{sl}(3,\mathbb{C})$ *denoted* $\mathbb{W}_i$ *with highest weight* $(p+q+r-3i, 0)$ *with* $i = 0, \ldots, r-1$
- $\mathcal{S}_{p+i,p+i,r+q-p-2i}$*, which is the highest weight vector of the unique irreducible representation of* $\mathfrak{sl}(3,\mathbb{C})$ *denoted* $\mathbb{Z}_i$ *with highest weight* $(0, 2p-q-r+3i)$ *with* $i = 1, \ldots, \lfloor (r+q-p)/2 \rfloor$.

Proof. Fix i in its interval and let u_i, v_i, w_i and z_i be highest weight vectors in $\mathbb{U}_i$, $\mathbb{V}_i$, $\mathbb{W}_i$ and $\mathbb{Z}_i$ repectively. We will show that there exist a weight vector in each of these spaces with weight $(p-q, q-r)$ of multiplicity 1. Equivalently, the corresponding weight spaces have dimension 1:

$$
\begin{aligned}
\operatorname{span}\{(Y_1)^i u_i\} &\subset \mathbb{U}_i \\
\operatorname{span}\{(Y_2)^i v_i\} &\subset \mathbb{V}_i \\
\operatorname{span}\{(Y_1)^{q-r}(Y_1Y_3)^{r-i} w_i\} &\subset \mathbb{W}_i \\
\operatorname{span}\{(Y_2)^{p-q}(Y_2Y_3)^{i} z_i\} &\subset \mathbb{Z}_i.
\end{aligned}
$$

Using (2.47), it is easy to verify that these vectors correspond to the weight $(p-q, q-r)$. The fact that they occur with multiplicity 1 follows also from the pattern of multiplicities. Indeed, the highest weight vectors have multiplicity 1 and acting with Y_1 and Y_2 does not change that, because it implies moving along the outermost hexagon or triangle of the weights in $\mathbb{U}_i$, $\mathbb{V}_i$, $\mathbb{W}_i$ or $\mathbb{Z}_i$. Note that w_i and z_i lie on the boundaries of the Weyl chamber. Hence, the weights in $\mathbb{W}_i$ and $\mathbb{Z}_i$ will occur on triangles. The action with Y_1Y_3 and Y_2Y_3 implies moving to the inward triangles and the multiplicity does not change on triangles. $\square$

In Figure 2.5 the highest weights of the above lemma are visualised together in the Weyl chamber, which is a 60°-section.

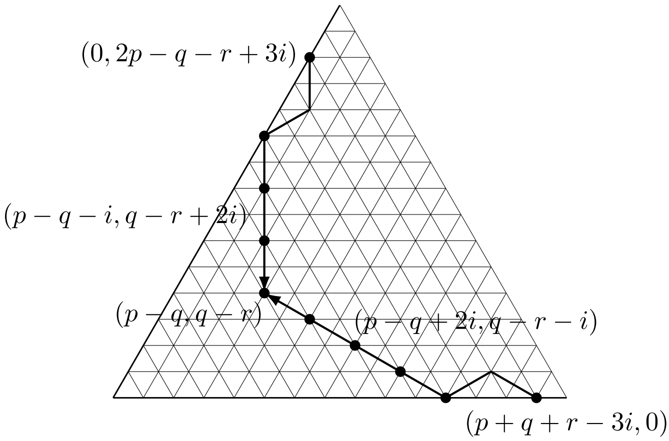


Figure 2.5: Weights of multiplicity 1.

It is also clear from the Clifford analysis point of view that these vector spaces have multiplicity 1 in the decomposition of $\mathcal{M}_{p,q,r}$. Because we have that $[\langle u, \partial_x\rangle, \langle v, \partial_x\rangle] = 0$ and $[\langle v, \partial_u\rangle, \langle v, \partial_x\rangle] = 0$, the only possible embeddings are given in the next lemma.

Lemma 4. *The following spaces occur with multiplicity 1 in the decomposition of $\mathcal{M}_{p,q,r}$:*

- $\langle u, \partial_x\rangle^i \mathcal{S}_{p+i,q-i,r}$ *for* $i = 0, \ldots, q-r$
- $\langle v, \partial_u\rangle^i \mathcal{S}_{p,q+i,r-i}$ *for* $i = 1, \ldots, \min(r, p-q)$
- $\langle v, \partial_x\rangle^{r-i} \langle u, \partial_x\rangle^{q-i} \mathcal{S}_{p+q+r-2i,i,i}$ *for* $i = 0, \ldots, r-1$
- $\langle v, \partial_x\rangle^i \langle v, \partial_u\rangle^{p-q+i} \mathcal{S}_{p+i,p+i,r+q-p-2i}$ *for* $i = 1, \ldots, \lfloor (r+q-p)/2 \rfloor$.

Indeed, each space is a subspace of $\mathcal{M}_{p,q,r}$: they have the correct degree of homogeneity and they are null solutions of the Dirac operators.

As an example, we will explicitly prove the fourth result in the above lemma:

$$\langle v, \partial_x \rangle^i \langle v, \partial_u \rangle^{p-q+i} : \mathcal{S}_{p+i,p+i,r+q-p-2i} \hookrightarrow \mathcal{M}_{p,q,r}.$$

This follows from the definition of simplicial monogenics in combination with the identities

$$\begin{aligned}
[\partial_x, \langle v, \partial_x \rangle^i \langle v, \partial_u \rangle^{p-q+i}] &= 0 \\
[\partial_u, \langle v, \partial_x \rangle^i \langle v, \partial_u \rangle^{p-q+i}] &= 0 \\
[\partial_v, \langle v, \partial_x \rangle^i] &= i \langle v, \partial_x \rangle^{i-1} \partial_x \\
[\partial_v, \langle v, \partial_u \rangle^{p-q+i}] &= (p-q+i) \langle v, \partial_u \rangle^{p-q+i-1} \partial_u.
\end{aligned}$$

The remaining embedding factors are proved analogously.

Next, we deal with the multiplicities higher than 1 by slightly adapting the previous situation.

Lemma 5. *The following vector spaces have multiplicity $j+1$ $(j > 0)$ in the decomposition of $\mathcal{M}_{p,q,r}$:*

- $\mathcal{S}_{p+i+j,q-i,r-j}$, *which is the highest weight vector of the unique irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ denoted $\mathbb{U}_{i,j}$ with highest weight $(p-q+2i+j, q-r-i+j)$ with $i = 0, \ldots, q-r$ and $j = 1, \ldots, r$*
- $\mathcal{S}_{p+i+j,q-i,r-j}$, *which is the highest weight vector of the unique irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ denoted $\mathbb{V}_{i,j}$ with highest weight $(p-q-i+j, q-r+2i+j)$ with $i = 1, \ldots, \min(r, p-q)$ and $j = 1, \ldots, r-i$*
- $\mathcal{S}_{p+q+r-2i+j,i,i-j}$, *which is the highest weight vector of the unique irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ denoted $\mathbb{W}_{i,j}$ with highest weight $(p+q+r-3i+j, j)$ with $i = 0, \ldots, r-1$ and $j = 1, \ldots, i$*
- $\mathcal{S}_{p+i+j,p+i,r+q-p-2i-j}$, *which is the highest weight vector of the unique irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ denoted $\mathbb{Z}_{i,j}$ with highest weight $(j, 2p-q-r+3i+j)$ with $i = 1, \ldots, \lfloor (r+q-p)/2 \rfloor$ and $j = 1, \ldots, r+q-p-2i$.*

Proof. Fix i and j in their respective intervals and let $u_{i,j}$, $v_{i,j}$, $w_{i,j}$ and $z_{i,j}$ be highest weight vectors in $\mathbb{U}_{i,j}$, $\mathbb{V}_{i,j}$, $\mathbb{W}_{i,j}$ and $\mathbb{Z}_{i,j}$ repectively. We will show that there is a weight vector in each of these spaces with weight $(p-q, q-r)$. The dimension of these weight spaces is $j+1$. Explicitly,

$$\begin{aligned}
\operatorname{span}\{(Y_3)^k(Y_1Y_2)^{j-k}(Y_1)^i u_{i,j} \,|\, k \in [0,j]\} &\subset \mathbb{U}_{i,j}\\
\operatorname{span}\{(Y_3)^k(Y_1Y_2)^{j-k}(Y_2)^i v_{i,j} \,|\, k \in [0,j]\} &\subset \mathbb{V}_{i,j}\\
\operatorname{span}\{(Y_3)^k(Y_1Y_2)^{j-k}(Y_1)^{q-r}(Y_1Y_3)^{r-i} w_{i,j} \,|\, k \in [0,j]\} &\subset \mathbb{W}_{i,j}\\
\operatorname{span}\{(Y_3)^k(Y_1Y_2)^{j-k}(Y_2)^{p-q}(Y_2Y_3)^{i} z_{i,j} \,|\, k \in [0,j]\} &\subset \mathbb{Z}_{i,j}.
\end{aligned}$$

Again, using (2.47), it is easily verified that every vector in these weight spaces has weight $(p-q, q-r)$. The fact that they have multiplicity $j+1$ follows also from the pattern of multiplicities: the operator $(Y_3)^k(Y_1Y_2)^{j-k}$ moves inward one ring. This corresponds with $j+1$ different 'paths'. Figure 2.6 shows 5 different paths for $j=4$. $\square$

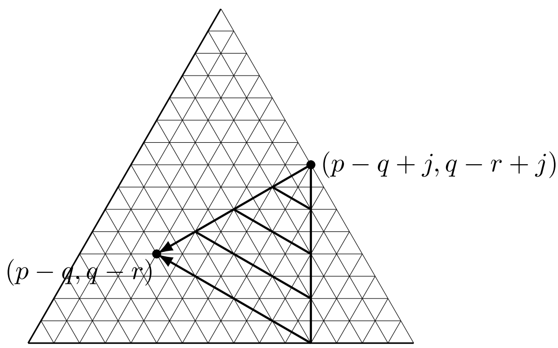


Figure 2.6: Multiplicities higher than 1.

For example, $\mathcal{S}_{p+2,q,r-2}$ occurs with multiplicity 3 in the decomposition of $\mathcal{M}_{p,q,r}$. Indeed, we have $\langle v, \partial_x\rangle^2\mathcal{S}_{p+2,q,r-2}$, $\langle v, \partial_x\rangle\langle u, \partial_x\rangle\langle v, \partial_u\rangle\mathcal{S}_{p+2,q,r-2}$, and $\big[\langle u, \partial_x\rangle\langle v, \partial_u\rangle\big]^2\mathcal{S}_{p+2,q,r-2}$, which are subspaces of $\mathcal{M}_{p,q,r}$, and the respective embedding maps are linearly independent. In general, in order to find modules of multiplicity $j+1$ with $j \geq 1$, we act with the operators $\langle v, \partial_x\rangle^k\big[\langle u, \partial_x\rangle\langle v, \partial_u\rangle\big]^{j-k}$ $(0 \leq k \leq j)$ on each space of multiplicity 1 described above.

As a final example, Figure 2.7 gives information about the decomposition of $\mathcal{M}_{4,3,2}$. The vector spaces $\mathcal{S}_{4,3,2}$, $\mathcal{S}_{4,4,1}$, $\mathcal{S}_{5,2,2}$, $\mathcal{S}_{7,1,1}$ and $\mathcal{M}_9$ occur with multiplicity 1, the vector spaces $\mathcal{S}_{5,3,1}$, $\mathcal{S}_{6,2,1}$, $\mathcal{S}_{8,1}$ and $\mathcal{S}_{5,4}$ have multiplicity 2 and the vector spaces $\mathcal{S}_{7,2}$ and $\mathcal{S}_{6,3}$ have multiplicity 3. These highest weight vectors of representation of $\mathfrak{sl}(3,\mathbb{C})$ are denoted by their highest weight as an irreducible Spin(m)-module.

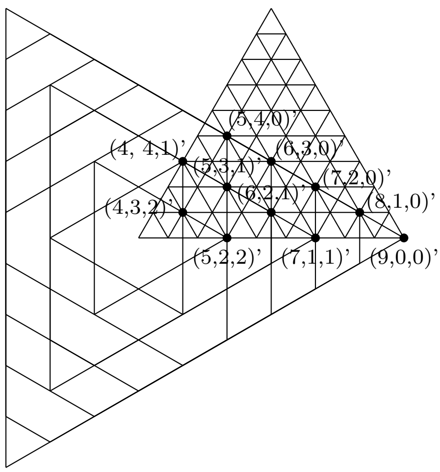


Figure 2.7: Decomposition of $\mathcal{M}_{4,3,2}$.

Summarising, we have

Proposition 6. *The decomposition into Spin(m)-irreducibles of the space of triple monogenics is given by*

$$
\begin{aligned}
\mathcal{M}_{p,q,r} &= \bigoplus_{i=0}^{q-r}\bigoplus_{j=0}^{r}\bigoplus_{k=0}^{j}\langle v,\partial_x\rangle^k\big[\langle u,\partial_x\rangle\langle v,\partial_u\rangle\big]^{j-k}\langle u,\partial_x\rangle^i\mathcal{S}_{p+i+j,q-i,r-j}\\
&\oplus\bigoplus_{i=0}^{r-1}\bigoplus_{j=0}^{i}\bigoplus_{k=0}^{j}\langle v,\partial_x\rangle^k\big[\langle u,\partial_x\rangle\langle v,\partial_u\rangle\big]^{j-k}\langle v,\partial_x\rangle^{r-i}\langle u,\partial_x\rangle^{q-i}\\
&\qquad\qquad\qquad\qquad\cdot\mathcal{S}_{p+q+r+j-2i,i,i-j}\\
&\oplus\bigoplus_{i=1}^{\min(r,p-q)}\bigoplus_{j=0}^{r-i}\bigoplus_{k=0}^{j}\langle v,\partial_x\rangle^k\big[\langle u,\partial_x\rangle\langle v,\partial_u\rangle\big]^{j-k}\langle v,\partial_u\rangle^i\mathcal{S}_{p+j,q+i,r-i-j}\\
&\oplus\bigoplus_{i=1}^{\lfloor(p+q-r)/2\rfloor}\bigoplus_{j=0}^{p+q-r-2i}\bigoplus_{k=0}^{j}\langle v,\partial_x\rangle^k\big[\langle u,\partial_x\rangle\langle v,\partial_u\rangle\big]^{j-k}\langle v,\partial_x\rangle^i\\
&\qquad\qquad\qquad\qquad\cdot\langle v,\partial_u\rangle^{p-q+i}\mathcal{S}_{p+i+j,p+i,r+q-p-2i-j}.
\end{aligned}
$$

In this way the following dimension formula for the triple monogenics is obtained:

Proposition 7. *The dimension formula for triple monogenics is given by*

$$
\begin{aligned}
\dim\mathcal{M}_{p,q,r} &= \sum_{i=0}^{q-r}\sum_{j=0}^{r}(j+1)\dim\mathcal{S}_{p+i+j,q-i,r-j}\\
&+\sum_{i=0}^{r-1}\sum_{j=0}^{i}(j+1)\dim\mathcal{S}_{p+q+r+j-2i,i,i-j}\\
&+\sum_{i=1}^{\min(r,p-q)}\sum_{j=0}^{r-i}(j+1)\dim\mathcal{S}_{p+j,q+i,r-i-j}\\
&+\sum_{i=1}^{\lfloor(r+q-p)/2\rfloor}\sum_{j=0}^{r+q-p-2i}(j+1)\dim\mathcal{S}_{p+i+j,p+i,r+q-p-2i-j}.
\end{aligned}\tag{2.48}
$$

2.4.3 Special triple monogenics

The case $h \geq k$

Let $h \geq k \geq l$. In what follows, we will frequently use the vector space

$$\mathcal{M}^s_{h,k,l} = \{f \in \mathcal{M}_{h,k,l}(\mathbb{R}^{3m}, \mathbb{S}) \,|\, \langle u, \partial_v \rangle f = 0\}. \tag{2.49}$$

In order to describe this space algebraically, branching rules from $\mathfrak{gl}(3,\mathbb{C})$ to $\mathfrak{gl}(2,\mathbb{C})$ are used, see e.g. [54]. Referring to [28] for a general theorem about the decomposition of spaces of monogenics, we have the following decomposition of $\mathcal{M}^s_{h,k,l}$ into irreducible modules, denoted by their highest weight with respect to the regular Spin group representation:

$$\mathcal{M}^s_{h,k,l} \cong \bigoplus_{\lambda} (\lambda_0, \lambda_1, \lambda_2)' = \bigoplus_{a,b} (h + a + b, k - a, l - b)' \tag{2.50}$$

with λ_i and a, b positive integers satisfying

$$\lambda_0 - \lambda_1 \geq a \geq 0, \quad \lambda_1 - \lambda_2 \geq b \geq 0. \tag{2.51}$$

Taking into account that $(h + a + b, k - a, l - b)'$ has to be a dominant weight with a and b positive integers, we find

$$\max(0, k - l - h) \leq a \leq k - l \tag{2.52}$$

$$\max(0, k - h - a) \leq b \leq l. \tag{2.53}$$

If $h \geq k$, we have $0 \leq a \leq k - l$ and $0 \leq b \leq l$. Therefore, the following modules can be embedded in $\mathcal{M}^s_{h,k,l}$:

$$\mathcal{S}_{h+i+j,k-i,l-j}$$

for all $0 \leq i \leq k - l$ and $0 \leq j \leq l$. Having determined the irreducible modules in the decomposition of $\mathcal{M}^s_{h,k,l}$, it is not difficult to find the corresponding embedding maps:

$$\langle u, \partial_x \rangle^i : \mathcal{S}_{h+i,k-i,l} \hookrightarrow \mathcal{M}^s_{h,k,l} \tag{2.54}$$

$$\widetilde{\langle v, \partial_x \rangle}^j := \big(\langle v, \partial_x \rangle (\mathbb{E}_u - \mathbb{E}_v) - \langle u, \partial_x \rangle \langle v, \partial_u \rangle\big)^j : \mathcal{S}_{h+j,k,l-j} \hookrightarrow \mathcal{M}^s_{h,k,l}. \tag{2.55}$$

Indeed, since

$$\begin{aligned}
[\partial_x, \langle u, \partial_x\rangle] &= 0 & [\partial_x, \langle \widetilde{v, \partial_x}\rangle] &= 0\\
[\partial_u, \langle u, \partial_x\rangle] &= \partial_x & [\partial_u, \langle \widetilde{v, \partial_x}\rangle] &= \langle v, \partial_x\rangle\partial_u - \langle v, \partial_u\rangle\partial_x\\
[\partial_v, \langle u, \partial_x\rangle] &= 0 & [\partial_v, \langle \widetilde{v, \partial_x}\rangle] &= (\mathbb{E}_u - \mathbb{E}_v)\partial_x - \langle v, \partial_x\rangle\partial_v - \langle u, \partial_x\rangle\partial_u\\
[\langle u, \partial_v\rangle, \langle u, \partial_x\rangle] &= 0 & [\langle u, \partial_v\rangle, \langle \widetilde{v, \partial_x}\rangle] &= 0,
\end{aligned}$$

it is obvious that they map simplicial monogenics to polynomials in $\mathcal{M}^s_{h,k,l}$. Furthermore, it follows from a straightforward calculation that the embedding maps commute:

$$[\langle u, \partial_x\rangle, \langle \widetilde{v, \partial_x}\rangle] = 0. \tag{2.56}$$

This means that every irreducible Spin(m)-module occurs with multiplicity 1 in the decomposition of $\mathcal{M}^s_{h,k,l}$, which is a general fact for branching rules from $\mathfrak{gl}(m, \mathbb{C})$ to $\mathfrak{gl}(m-1, \mathbb{C})$. Summarising, we have

$$\mathcal{M}^s_{h,k,l} = \bigoplus_{i=0}^{k-l}\bigoplus_{j=0}^{l} \langle u, \partial_x\rangle^i \langle \widetilde{v, \partial_x}\rangle^j \mathcal{S}_{h+i+j,k-i,l-j} \tag{2.57}$$

from which

$$\dim \mathcal{M}^s_{h,k,l} = \sum_{i=0}^{k-l}\sum_{j=0}^{l} \dim \mathcal{S}_{h+i+j,k-i,l-j}. \tag{2.58}$$

In chapter 8 we prove an alternative expression for the dimension of $\mathcal{M}^s_{h,k,l}$.

The case $h < k$

Which summands occur in the decomposition of $\mathcal{M}^s_{h,k,l}$ when $k \geq l$ and $h < k$? One might think it is suffices to consider (2.57) and then omit all summands that do not satisfy the highest weight condition. It appears the situation is more subtle than that. For example, let us decompose $\mathcal{M}^s_{1,4,2}$ into Spin(m)-irreducibles. It follows from (2.57) that, if $h \geq 4$, we have

$$\begin{aligned}
\mathcal{M}^s_{h,4,2} \cong{}& (h,4,2)' \oplus (h+1,3,2)' \oplus (h+2,2,2)' \oplus (h+1,4,1)' \oplus (h+2,3,1)'\\
&\oplus (h+3,2,1)' \oplus (h+2,4,0)' \oplus (h+3,3,0)' \oplus (h+4,2,0)'.
\end{aligned} \tag{2.59}$$

Substituting $h = 1$ in the above expression and omitting all summands that do not correspond to a highest weight, leads to

$$\mathcal{M}^s_{1,4,2} \cong (3,2,2)' \oplus (3,3,1)' \oplus (4,2,1)' \oplus (4,3,0)' \oplus (5,2,0)'.$$

However, experimenting with LiE ([50]) revealed that

$$\mathcal{M}^s_{1,4,2} \cong (4,3,0)' \oplus (4,2,1)' \oplus (5,2,0)'. \tag{2.60}$$

We refer to section 8.2 for more information about experiments with the computer algebra package for Lie group computations LiE. The result (2.60) can be explained by the branching rules of the previous section. As we are not dealing with a dominant weight $(h,k,l)'$, it thus follows from (2.50), (2.52) and (2.53) that

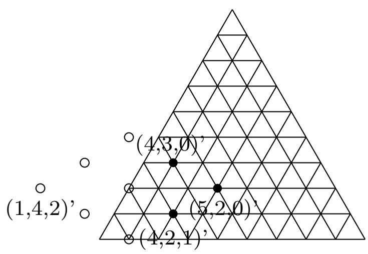
Proposition 8. *For all integers h, k and l, one has*

$$\mathcal{M}^s_{h,k,l} \cong \bigoplus_{a=\max(0,k-l-h)}^{k-l} \bigoplus_{b=\max(0,k-h-a)}^{l} (h+a+b, k-a, l-b)'. \tag{2.61}$$

Applying this proposition on the above example, we find

$$\begin{aligned}\mathcal{M}^s_{1,4,2} &\cong \bigoplus_{a=1}^{2} \bigoplus_{b=\max(0,3-a)}^{2} (1+a+b, 4-a, 2-b)' \\ &= (4,3,0)' \oplus (4,2,1)' \oplus (5,2,0)'\end{aligned}$$

which is in correspondence with (2.60). This can be visualised by means of a Weyl chamber, where the modules that occur in the decomposition of $\mathcal{M}^s_{1,4,2}$ are visualised by a black dot, while the extra modules that are obtained by putting $h=1$ in the formula (2.59), are visualised by a circle.



We illustrate this proposition with two more examples.

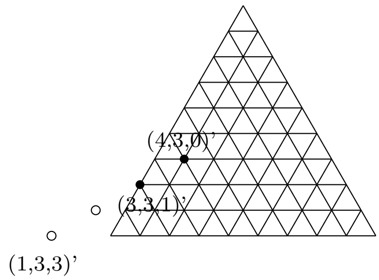
1. Consider the decomposition

$$\mathcal{M}^s_{1,3,3} \cong \bigoplus_{a=0}^{0} \bigoplus_{b=\max(0,2-a)}^{3} (1+a+b, 3-a, 3-b)' = (3,3,1)' \oplus (4,3,0)'.$$

For $h \geq 3$, we have

$$\mathcal{M}^s_{h,3,3} \cong (h,3,3)' \oplus (h+1,3,2)' \oplus (h+2,3,1)' \oplus (h+3,3,0)'. \qquad (2.62)$$

With the same remark as before, this can be visualised as



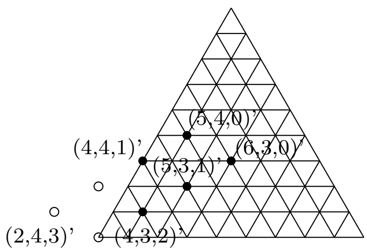
2. Consider the decomposition

$$\mathcal{M}^s_{2,4,3} \cong \bigoplus_{a=0}^{1} \bigoplus_{b=\max(0,2-a)}^{3} (2+a+b, 4-a, 3-b)'$$
$$\cong (4,4,1)' \oplus (5,4,0)' \oplus (4,3,2)' \oplus (5,3,1)' \oplus (6,3,0)'.$$

For $h \geq 4$, we have

$$\mathcal{M}^s_{h,4,3} \cong (h,4,3)' \oplus (h+1,3,3)' \oplus (h+1,4,2)' \oplus (h+2,3,2)'$$
$$\oplus (h+2,4,1)' \oplus (h+3,3,1)' \oplus (h+3,4,0)' \oplus (h+4,3,0)'.$$

These modules can be visualised as follows:



Note once again that finding the decomposition of the 'degenerate' case $h < k$ is more subtle than to just take the intersection with the Weyl chamber. Not every dominant weight, as one might expect from the decomposition (2.57), occurs in the decomposition (2.61).

2.4.4 Other spaces of monogenics

The next results will be used in chapter 5. Consider the notation (2.43) with $k > 1$. Applying once more the branching theorem in [28], we conclude that

$$\mathcal{M}^s_{\lambda_{h,k}} = \{f \in \mathcal{M}_{\lambda_{h,k}} \,|\, \langle u_i, \partial_j \rangle f = 0 \ , \ 1 \leq i \neq j \leq k\} \tag{2.63}$$

has the decomposition

$$\mathcal{M}^s_{\lambda_{h,k}} \cong \mathcal{S}_{\lambda_{h,k}} \oplus \mathcal{S}_{\lambda_{h+1,k-1}}. \tag{2.64}$$

It is not difficult to show that an embedding map for $\mathcal{S}_{\lambda_{h+1,k-1}}$ is given by

$$\left(\langle u_k, \partial_x \rangle - \sum_{i=1}^{k-1} \langle u_i, \partial_x \rangle \langle u_k, \partial_i \rangle \right) : \mathcal{S}_{\lambda_{h+1,k-1}} \hookrightarrow \mathcal{M}_{\lambda_{h,k}}.$$

Chapter 3

(Higher spin) Dirac operators

In section 2.2 it was mentioned that the (standard) Dirac operator ∂_x on $\mathbb{R}^m$, defined as

$$\partial_x = \sum_{j=1}^{m} e_j \partial_{x_j},$$

plays a fundamental role in Clifford analysis, see e.g. [4, 30, 68]. This operator is an elliptic Spin(m)-invariant, even conformally invariant, first-order differential operator acting on $\mathbb{S}$-valued functions:

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}).$$

The Dirac operator is not only studied in Clifford analysis; it is an important topic in representation theory and theoretical physics as well. It follows from results in [15, 38, 65] that there exists a 'sequence' of similar elliptic conformally invariant first-order differential operators acting on functions with values in more complicated representations $\mathbb{V}_\lambda$ of Spin(m) (see also Theorem 15 of this chapter). These operators are called *higher spin Dirac operators* and the (standard) Dirac operator can be considered as the easiest case in this sequence. In the next chapters, we will discuss three higher spin Dirac operators acting on functions in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$: in chapter 4 we have $\mathbb{V}_\lambda = \mathcal{M}_k$, in chapter 5 we consider $\mathbb{V}_\lambda = \mathcal{S}_{\lambda_k}$ and the operator playing the main role in chapter 6 (and the subsequent chapters) acts on functions in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$.

As we will generalise properties of the standard Dirac operator in the next chapters, it is obvious that ∂_x deserves a chapter of its own. We have mentioned several times that the Dirac operator is elliptic and Spin(m)-invariant; we will prove these statements in section 3.1. Several important results in Clifford analysis, such as the Fischer decomposition, the CK-extension, the fundamental solution and basic integral formulae, are given in section 3.2. We conclude this chapter with the definition and some properties of higher spin Dirac operators.

3.1 Properties of the standard Dirac operator

3.1.1 Ellipticity

Definition 3. *A differential operator $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ of order $k \in \mathbb{N}$ in $\Omega \subset \mathbb{R}^m$ is elliptic if for all $x \in \Omega$ and for all $\xi \neq 0$ in $\mathbb{R}^m$, the principle symbol is invertible, i.e. $\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \neq 0$.*

Since the principle symbol of the Dirac operator ∂_x is the vector variable x, the Dirac operator is elliptic.

3.1.2 Spin group invariance

In this section we will prove that the Dirac operator is Spin(m)-invariant. In the world of Clifford analysis, it is less well-known that this operator is even conformally invariant; this topic will be addressed in section 3.3.1. In order to prove the Spin(m)-invariance, we begin with the following result.

Lemma 6. *If y is a vector in $\mathbb{R}^m$, one has that $\partial_x = y\partial_{yxy}y$.*

Proof. This statement follows from the chain rule and the fact that

$$yxy = -2\langle x, y\rangle y + |y|^2 x.$$

Indeed,

$$\begin{aligned}
\partial_x = \sum_{j=1}^m e_j \partial_{x_j} &= \sum_{j=1}^m e_j \sum_{i=1}^m \partial_{(yxy)_i} \frac{\partial}{\partial x_j}(-2\langle x, y\rangle y_i + |y|^2 x_i) \\
&= \sum_{j=1}^m e_j \sum_{i=1}^m \partial_{(yxy)_i}(-2y_j y_i + |y|^2 \delta_{ij}) \\
&= -2\langle \partial_{yxy}, y\rangle y + |y|^2 \partial_{yxy} = y\partial_{yxy} y
\end{aligned}$$

which concludes the proof. □

In particular, if y is a unit vector, then $\overline{y} = -1$ and $y^{-1} = -y$. It follows from the above lemma that

$$y\partial_x y = \partial_{yxy}$$

for unit vectors y.

Lemma 7. *The Dirac operator ∂_x is Spin(m)-invariant.*

Proof. Let $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S})$ and $s \in \mathrm{Spin}(m)$, then it follows from Lemma 6 and (2.15) that

$$\begin{aligned} L(s)\partial_x f(x) &= s\partial_{\bar{s}xs} f(\bar{s}xs) \\ &= s\bar{s}\partial_x s f(\bar{s}xs) \\ &= \partial_x s f(\bar{s}xs) \\ &= \partial_x L(s) f(x). \end{aligned}$$

□

The Spin(m)-invariance of the Dirac operator can be visualised by means of a commuting diagram:

$$\begin{tikzcd} \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \arrow[r, "\partial_x"] \arrow[d, "L(s)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \arrow[d, "L(s)"] \\ \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \arrow[r, "\partial_x"'] & \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \end{tikzcd}$$

3.2 Results in Clifford analysis

For detailed information on the topic of the Dirac operator in Clifford analysis, we refer to the standard works [4, 30, 68].

3.2.1 Null solutions

We have shown in section 2.3.6 that the vector space

$$\mathcal{M}_k = \{f \in \mathcal{P}_k(\mathbb{R}^m, \mathbb{S}) \,|\, \partial_x f = 0\}$$

of spherical monogenics of degree k is an irreducible representation of Spin(m) with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ of the L-representation. It is important to point this out, because the vector space of homogeneous (of a fixed degree) null solutions of the higher spin Dirac operators that we will discuss in the next chapters, will no longer be irreducible with respect to the Spin group.

3.2.2 Fischer inner product

Before defining the Fischer inner product, we introduce some notations. If α denotes the multi-index $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m$, then

$$|\alpha| := \sum_{i=1}^{m} \alpha_i \qquad\qquad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$$

$$\alpha! := \alpha_1! \alpha_2! \cdots \alpha_m! \qquad\qquad \partial^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_m}^{\alpha_m}.$$

Clearly,

$$\partial^\alpha x^\beta = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \tag{3.1}$$

Let f and g be two polynomials in $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}_m) = \operatorname{span}_{\mathbb{C}_m}\{x^\alpha \,|\, |\alpha| = k\}$, i.e. $f(x) = \sum_{|\alpha|=k} x^\alpha a_\alpha$ and $g(x) = \sum_{|\beta|=k} x^\beta b_\beta$ with a_α, b_β in $\mathbb{C}_m$. Denote by $f(\partial)$ the differential operator obtained by replacing x_i by ∂_{x_i} in $f(x)$ with $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. The Fischer inner product of f and g is defined as

$$\langle f, g \rangle_k := \left[\sum_{|\alpha|=k} \alpha! \, \overline{a}_\alpha b_\alpha \right]_0 = \left[f^\dagger(\partial) g(x) \right]_0. \tag{3.2}$$

The second expression follows from (3.1). Applying this definition on the Fischer inner product $\langle xf, g \rangle_k$, with $f \in \mathcal{P}_{k-1}(\mathbb{R}^m, \mathbb{C}_m)$ and $g \in \mathcal{P}_k(\mathbb{R}^m, \mathbb{C}_m)$, we find

$$\langle xf, g \rangle_k = -\langle f, \partial_x g \rangle_{k-1}. \tag{3.3}$$

3.2.3 Fischer decomposition

A well-known theorem in harmonic analysis is the (harmonic) Fischer decomposition.

Theorem 9 (Harmonic Fischer decomposition)**.** *One has*

$$\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) = \mathcal{H}_k \oplus x^2 \mathcal{P}_{k-2}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{s=0}^{\lfloor k/2 \rfloor} x^{2s} \mathcal{H}_{k-2s}$$

and, more generally,

$$\mathcal{P}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{s=0}^{\infty} x^{2s} \mathcal{H}_k. \tag{3.4}$$

This theorem is visualised in Figure 3.1.

$$
\begin{tikzcd}
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 & \cdots \\
\parallel & \parallel & \parallel & \parallel & \parallel & \parallel & \\
\mathcal{H}_0 \arrow[rr] & & x^2\mathcal{H}_0 \arrow[rr] & & x^4\mathcal{H}_0 & & \cdots \\
& \mathcal{H}_1 \arrow[rr] & \oplus & x^2\mathcal{H}_1 \arrow[rr] & \oplus & x^4\mathcal{H}_1 & \\
& & \mathcal{H}_2 \arrow[rr] & \oplus & x^2\mathcal{H}_2 & \oplus & \cdots \\
& & & \mathcal{H}_3 \arrow[rr] & \oplus & x^2\mathcal{H}_3 & \\
& & & & \mathcal{H}_4 & \oplus & \cdots \\
& & & & & \mathcal{H}_5 &
\end{tikzcd}
$$

Figure 3.1: Representation of the harmonic Fischer decomposition.

In the above figure, the kth column represents the decomposition of the space $\mathcal{P}_k$ into $SO(m)$-invariant (and irreducible) modules $x^{2s}\mathcal{H}_{k-2s}$; on the jth row there are infinitely many isomorphic copies of the space $\mathcal{H}_{j-1}$. In fact, each row corresponds to an infinite-dimensional representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \cong \text{span}\{\mathbb{E}_x + \frac{m}{2}, \frac{1}{2}(-x^2), \frac{1}{2}\Delta_x\}$. This hidden symmetry in (3.4) can be summarised by saying that the Howe-dual pair (see [47, 48]) for this decomposition is given by $(SO(m), \mathfrak{sl}(2,\mathbb{R}))$. In general, the couple $(SO(m), \mathfrak{g})$ is called a Howe-dual pair, with respect to a bigger Lie algebra in which $\mathfrak{so}(m,\mathbb{R})$, i.e. the Lie algebra of $SO(m)$, and $\mathfrak{g}$ are commuting. It is not difficult to verify that the action (2.20) of $\mathfrak{so}(m,\mathbb{R})$ commutes with the operators $\mathbb{E}_x + \frac{m}{2}, \frac{1}{2}(-x^2), \frac{1}{2}\Delta_x$. Due to this approach, the Fischer decomposition can be written in a more abstract way. For more on Howe-duality in Clifford analysis, we refer to e.g. [9].

The result of Theorem 9 can be refined as follows. Applying (3.3) in the special case that $f \in \mathcal{P}_{k-1}(\mathbb{R}^m,\mathbb{C})$ and $g \in \mathcal{M}_k$, we have

$$\langle xf, g\rangle_k = 0.$$

This means that the following orthogonal decomposition holds:

Theorem 10 (Monogenic Fischer decomposition)**.** *One has*

$$\mathcal{P}_k(\mathbb{R}^m,\mathbb{C}) \otimes \mathbb{S} \,=\, \mathcal{M}_k \,\oplus\, (x\mathcal{P}_{k-1}(\mathbb{R}^m,\mathbb{C}) \otimes \mathbb{S}) \,=\, \bigoplus_{s=0}^{k} x^s\mathcal{M}_{k-s}$$

and more general,

$$\mathcal{P}(\mathbb{R}^m, \mathbb{S}) = \bigoplus_{k=0}^{\infty} \bigoplus_{s=0}^{\infty} x^s \mathcal{M}_k. \tag{3.5}$$

These decompositions are visualised in Figure 3.2.

$$\begin{tikzcd}
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \cdots \\
\| & \| & \| & \| & \| & \\
\mathcal{M}_0 \arrow[r] & x\mathcal{M}_0 \arrow[r] & x^2\mathcal{M}_0 \arrow[r] & x^3\mathcal{M}_0 \arrow[r] & x^4\mathcal{M}_0 & \cdots \\
 & \oplus & \oplus & \oplus & \oplus & \\
 & \mathcal{M}_1 \arrow[r] & x\mathcal{M}_1 \arrow[r] & x^2\mathcal{M}_1 \arrow[r] & x^3\mathcal{M}_1 & \cdots \\
 & & \oplus & \oplus & \oplus & \\
 & & \mathcal{M}_2 \arrow[r] & x\mathcal{M}_2 \arrow[r] & x^2\mathcal{M}_2 & \cdots \\
 & & & \oplus & \oplus & \\
 & & & \mathcal{M}_3 \arrow[r] & x\mathcal{M}_3 & \cdots \\
 & & & & \oplus & \\
 & & & & \mathcal{M}_4 & \cdots
\end{tikzcd}$$

Figure 3.2: Representation of the refined Fischer decomposition.

The decomposition (3.5) is clearly a refinement of the harmonic Fischer decomposition. The Howe-dual pair is $(\mathrm{Pin}(m), \mathfrak{osp}(1|2))$. Indeed, this follows from Theorem 1 and the fact that, on the one hand, the modules $x^s\mathcal{M}_k$ are invariant (and irreducible) under the action of $\mathrm{Pin}(m)$, and that, on the other hand, the action of $\mathfrak{so}(m, \mathbb{R})$ commutes with the operators of $\mathfrak{osp}(1|2)$.

It follows from the two above theorems that

Theorem 11 (Monogenic Fischer decomposition of harmonics)**.**

$$\mathcal{H}_k \otimes \mathbb{S} = \mathcal{M}_k \oplus x\mathcal{M}_{k-1}. \tag{3.6}$$

This is the version of the Fischer decomposition that is generalised in this thesis. It can be proved (see e.g. [30]) that for $f \in \mathcal{P}_{k-1}(\mathbb{R}^m, \mathbb{S})$, we have

$$xf \in \mathcal{H}_k \otimes \mathbb{S} \ \Leftrightarrow \ f \in \mathcal{M}_{k-1}. \tag{3.7}$$

It follows from (3.4) that

$$\mathcal{P}(\mathbb{R}^m, \mathbb{S}) = \bigoplus_{k=0}^{\infty} \bigoplus_{s=0}^{\infty} x^{2s} \mathcal{M}_k \oplus \bigoplus_{k=0}^{\infty} \bigoplus_{s=0}^{\infty} x^{2s+1} \mathcal{M}_k \tag{3.8}$$

and this decomposition corresponds to the Howe-dual pair $(\mathrm{Spin}(m), \mathfrak{osp}(1|2))$.

3.2.4 CK-extension

One way of constructing monogenic functions is by extending real-analytic functions in some open connected domain Ω^* in $\mathbb{R}^{m-1}$. The problem is the following: "Given a real-analytic function f^* in $\Omega^* \subset \mathbb{R}^{m-1}$, does there exist a monogenic function f in an open neighbourhood Ω of Ω^* in $\mathbb{R}^m$ such that $f|_{\Omega^*} = f^*$?"

Consider $\mathbb{R}^{m-1}$ as the hyperplane $x_m = 0$ in $\mathbb{R}^m$. The variable we use in $\mathbb{R}^{m-1}$ is $x^* = (x_1, \ldots, x_{m-1})$ and the Dirac operator in $\mathbb{R}^{m-1}$ is given by $\partial_{x^*} = \sum_{i=1}^{m-1} e_i \partial_{x_i}$. Let Ω be an open connected and x_m-normal neighbourhood of Ω^*, which means that for each $x \in \Omega$, the line segment $\{x + te_m \,|\, t \in \mathbb{R}\} \cap \Omega$ is connected and contains exactly one point in Ω^*. The existence of Ω can be proved (see e.g. [4]). The function f has to satisfy the conditions $\partial_x f = 0$ in Ω and $f(x^*, x_m)|_{x_m=0} = f(x^*, 0) = f^*(x^*)$. We have that

$$\partial_x f = 0 \Leftrightarrow \partial_{x_m} f = -\overline{e}_m \partial_{x^*} f$$

whence

$$f(x^*, x_m) = (e^{-x_m \overline{e}_m \partial_{x^*}}) f^*(x^*).$$

The existence of the monogenic function f in Ω is thus garantueed; it is called the *Cauchy-Kowalewskaia (CK) extension* of f^*. It is unique by construction.

In particular, the CK-extension allows for calculating the dimension of the vector space $\mathcal{M}_k$, which we have already found by means of the Weyl dimension formula in (2.39). If $p_k^* \in \mathcal{P}_k(\mathbb{R}^{m-1}, \mathbb{S})$, then the CK-extension p_k in $\mathbb{R}^m$ of p_k^* is given by

$$p_k(x) = \sum_{j=0}^{k} \frac{(-x_m)^j}{j!} (\overline{e}_m \partial_{x^*})^j p_k^*(x^*)$$

and, by definition, $p_k \in \mathcal{M}_k$. Conversely, if p_k is in $\mathcal{M}_k$, its restriction $q_k(x^*) := p_k(x^*, 0)$ to $\mathbb{R}^{m-1}$ is a homogeneous polynomial of degree k and the CK-extension of q_k is precisely p_k. The CK-extension thus establishes an isomorphism between $\mathcal{P}_k(\mathbb{R}^{m-1}, \mathbb{S})$ and $\mathcal{M}_k$. Hence

$$\dim \mathcal{M}_k = \dim \mathcal{P}_k(\mathbb{R}^{m-1}) \dim \mathbb{S} = 2^n \begin{pmatrix} k+m-2 \\ k \end{pmatrix} \tag{3.9}$$

which equals (2.39).

3.2.5 Fundamental solution

The fundamental solution of the Dirac operator is given by the Cauchy kernel $E(x)$, defined as

$$E(x) = -\frac{1}{A_m} \frac{x}{|x|^m}$$

where A_m denotes the surface area of the unit sphere in $\mathbb{R}^m$, i.e.

$$A_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}.$$

The Cauchy kernel satisfies in distributional sense,

$$\partial_x E(x) = \delta(x).$$

The distribution δ in $\mathbb{R}^m$ acts as follows on the vector space $\mathcal{D}(\mathbb{R}^m, \mathbb{C}_m)$ of test functions in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{C}_m)$ with compact support:

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0)$$

for all φ in $\mathcal{D}(\mathbb{R}^m, \mathbb{C}_m)$. This Cauchy kernel was essential in the development of Euclidean Clifford analysis (see e.g. [4]).

3.2.6 Basic integral formulae

We introduce the volume element $dx = dx_1 \wedge \ldots \wedge dx_m$ and the oriented surface element $d\sigma_x = \sum_{j=1}^m (-1)^{j-1} e_j d\widehat{x}_j$ with $d\widehat{x}_j = dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$. For the proofs of the next theorems, we refer to [4].

Theorem 12 (Stokes)**.** *Let $\Omega' \subset \mathbb{R}^m$ and $\overline{\Omega} \subset \Omega'$ and let $f, g \in \mathcal{C}^1(\Omega', \mathbb{C}_m)$. One has*

$$\int_\Omega \left[(f\partial_x)g + f(\partial_x g) \right] dx = \int_{\partial\Omega} f(x) d\sigma_x g(x).$$

Theorem 13 (Cauchy-Pompeiu). *Let $\Omega' \subset \mathbb{R}^m$ and $\overline{\Omega} \subset \Omega'$ and let $f \in \mathcal{C}^1(\Omega', \mathbb{C}_m)$. Then,*

$$-\int_\Omega E(x-y)\partial_x f(x)dx + \int_{\partial\Omega} E(x-y)d\sigma_x f(x) = \begin{cases} f(y) & \forall y \in \Omega \\ 0 & \forall y \in \Omega' \setminus \overline{\Omega} \end{cases}$$

Theorem 14 (Cauchy integral formula). *Let $\Omega' \subset \mathbb{R}^m$ and $\overline{\Omega} \subset \Omega'$ and let $f \in \mathcal{C}^1(\Omega', \mathbb{C}_m)$. If $\partial_x f = 0$ in Ω', then*

$$\int_{\partial\Omega} E(x-y)d\sigma_x f(x) = \begin{cases} f(y) & \forall y \in \Omega \\ 0 & \forall y \in \Omega' \setminus \overline{\Omega} \end{cases}$$

3.3 Higher spin Dirac operators

As mentioned in the beginning of this chapter, the standard Dirac operator is a special case of a sequence of the following elliptic conformally invariant first-order differential operators:

Theorem 15 (Adapted from [15, 38]). *For every Spin(m)-irreducible representation $\mathbb{V}_\lambda$ with highest weight $\lambda = (\lambda_1 + \frac{1}{2}, \ldots, \lambda_{n-1} + \frac{1}{2}, \frac{1}{2})$, there exists a unique (up to a to a multiplicative constant) elliptic conformally invariant first-order differential operator, on between functions taking values in $\mathbb{V}_\lambda$.*

In case $\lambda_1 = \ldots = \lambda_{n-1} = 0$, the irreducible representation of Spin(m) is the spinor space $\mathbb{V}_\lambda = \mathbb{S}$. Here is an overview of the higher spin Dirac operators that we will discuss in the next chapters:

- In case $\lambda_1 = k$ and $\lambda_2 = \ldots = \lambda_{n-1} = 0$, we know that $\mathbb{V}_\lambda = \mathcal{M}_k$ and the corresponding higher spin Dirac operator (also known as Rarita-Schwinger operator) is denoted by $\mathcal{R}_k$. See chapter 4 for more information.

- In chapter 5 we study the higher spin Dirac operator $\mathcal{Q}_{\lambda_k}$, which acts on functions with values in $\mathbb{V}_\lambda = \mathcal{S}_{\lambda_k}$, i.e. $\lambda_1 = \ldots = \lambda_k = 1$ and $\lambda_{k+1} = \ldots = \lambda_{n-1} = 0$ for $k < n-1$.

- Next in line behind the Rarita-Schwinger operator is the higher spin Dirac operator denoted by $\mathcal{Q}_{k,l}$, which is constructed in chapter 6 and plays the leading role in this thesis. In this case, we have $\mathbb{V}_\lambda = \mathcal{S}_{k,l}$, which corresponds by putting $\lambda_1 = k$, $\lambda_2 = l$ ($k \geq l$) and $\lambda_3 = \ldots = \lambda_{n-1} = 0$ in Theorem 15.

3.3.1 Conformal invariance

The Dirac operator ∂_x and higher spin Dirac operators are not just Spin(m)-invariant, they are also conformally invariant. The conformal invariance of these operators in Theorem 15 follows from a theorem in [38], which we have adapted to the present notations and operators.

Let $n = \lfloor \frac{m}{2} \rfloor$. Denote by μ_i the n-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 occurs in the ith entry ($1 \leq i \leq n$). Let Λ be the set of highest weights in the fundamental Weyl chamber; Λ is given by (2.30) or (2.32) in case $m = 2n$ or $m = 2n+1$, respectively. Furthermore, $\mathbb{W}_\rho$ denotes a (not necessarily irreducible) Spin(m)-module with highest weight ρ.

Theorem 16 ([38]). *If $\mathbb{R}^m \otimes \mathbb{W}_\rho = \sum_\lambda \mathbb{V}_\lambda$ is a decomposition into irreducible Spin(m)-modules, then each of these irreducible representations occurs with multiplicity one; the set Λ of highest weights λ is formed as follows:*

- *if $m = 2n$ then $\lambda \in \Lambda \Leftrightarrow \lambda = \rho \pm \mu_i$ $(1 \leq i \leq n)$*
- *if $m = 2n+1$ and $\lambda_n \geq \frac{1}{2}$, then we have*
  $\lambda \in \Lambda \Leftrightarrow$ either $\lambda = \rho$ or $\lambda = \rho \pm \mu_i$ $(1 \leq i \leq n)$
- *if $m = 2n+1$ and $\lambda_n = 0$, then we have*
  $\lambda \in \Lambda \Leftrightarrow$ either $\lambda = \rho + \mu_n$ or $\lambda = \rho \pm \mu_i$ $(1 \leq i < n)$.

This theorem implies that the decomposition $\mathbb{R}^m \otimes \mathbb{W}_\rho = \sum_\lambda \mathbb{V}_\lambda$ is unique.

We apply Theorem 16 on three cases (recall that we work in the odd-dimensional case $m = 2n+1$):

- if $\mathbb{W}_\rho = \mathbb{S}$, then
$$\Lambda = \{(\tfrac{3}{2}, \ldots, \tfrac{1}{2}), (\tfrac{1}{2}, \ldots, \tfrac{1}{2})\}$$
- if $\mathbb{W}_\rho = \mathcal{M}_k$, then
$$\begin{aligned}\Lambda =& \{(k+\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), (k+\tfrac{3}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}),\\ &(k-\tfrac{1}{2}, \ldots, \tfrac{1}{2}), (k+\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})\}\end{aligned}$$
- if $\mathbb{W}_\rho = \mathcal{S}_{k,l}$, then
$$\begin{aligned}\Lambda =& \{(k+\tfrac{1}{2}, l+\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), (k+\tfrac{3}{2}, l+\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}),\\ &(k-\tfrac{1}{2}, l+\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), (k+\tfrac{1}{2}, l+\tfrac{3}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}),\\ &(k+\tfrac{1}{2}, l-\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), (k+\tfrac{1}{2}, l+\tfrac{1}{2}, \tfrac{3}{2}, \ldots, \tfrac{1}{2})\}\end{aligned}$$

(on the condition that these weights are dominant.) In particular, this means that $\mathbb{S}$ occurs with multiplicity 1 in $\mathbb{R}^m \otimes \mathbb{S}$, and similarly for $\mathcal{M}_k$ and $\mathcal{S}_{k,l}$ in the decomposition of $\mathbb{R}^m \otimes \mathcal{M}_k$ and $\mathbb{R}^m \otimes \mathcal{S}_{k,l}$, respectively.

To formulate the next theorem, which is adapted from the main theorem in [38], let δ denote half the sum of the positive roots of $SO(m)$, i.e. $\delta = (n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2})$ and $-\frac{1}{2}||\cdot||^2 = \langle \cdot, \cdot \rangle$ from (2.22). We also introduce the projection operator $p_\lambda : \mathbb{R}^m \otimes \mathbb{W}_\rho \to \mathbb{V}_\lambda$ and the covariant derivative $\nabla : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{W}_\rho) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^m \otimes \mathbb{W}_\rho)$. Consider the same notations as before.

Theorem 17 (Adapted from [38])**.** *If the conformal weight w of the vector bundle associated to ρ equals*

$$w = \frac{1}{2}\left(m - 1 + ||\delta + \rho||^2 - ||\delta + \lambda||^2\right) \tag{3.10}$$

then the operator

$$\mathcal{D}_\lambda = p_\lambda \nabla : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{W}_\rho) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$$

is conformally invariant.

Let $\mathbb{W}_\rho$ be a Spin(m)-irreducible vector space, and let $\rho = \lambda$, i.e. $\mathbb{W}_\rho = \mathbb{V}_\lambda$. If the conformal weight of the vector bundle associated to the weight λ equals $w = \frac{m-1}{2}$, Theorem 17 states that the first-order differential operator

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$$

is conformally invariant. It can be proved (see [38, 67]) that the condition $w = \frac{m-1}{2}$ is fulfilled if λ equals one of the highest weights corresponding to an irreducible Spin(m)-module that we consider in this thesis: $\lambda = (\frac{1}{2}, \ldots, \frac{1}{2})$, $\lambda = (k+\frac{1}{2}, \ldots, \frac{1}{2})$, $\lambda = (k+\frac{1}{2}, l+\frac{1}{2}, \ldots, \frac{1}{2})$, which are precisely the highest weights of $\mathbb{S}$, $\mathcal{M}_k$ and $\mathcal{S}_{k,l}$, respectively.

Finally, we formulate a theorem in [65], which requires the following information. In order to describe the action of the conformal group $SO(m+1,1)$ in Clifford analysis, one considers the group Spin($m+1,1$), which is the double cover of the conformal group. Acting on $\mathbb{R}^m$, the group Spin($m+1,1$) can be described by means of the Vahlen group $V(m)$ (see e.g. [2, 22, 23, 55, 57]) of 2×2 matrices with $\mathbb{C}_m$-entries, i.e.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(m) \quad \Leftrightarrow \quad \begin{cases} a, b, c, d \in \Gamma(m) \cup \{0\} \\ a\widetilde{b}, c\widetilde{d}, \widetilde{d}b, \widetilde{c}a \in \mathbb{R}^m \\ a\widetilde{d} - b\widetilde{c} = \pm 1 \end{cases}$$

with $\Gamma(m)$ the Clifford group (2.1). It is a fact that all conformal transformations can be expressed in the form $\phi(x) = (ax+b)(cx+d)^{-1}$ where $x \in \mathbb{R}^m$ and $A \in V(m)$.

Now, let $\Omega \subset \mathbb{R}^m$ and consider the representation L of $\mathrm{Pin}(m)$ acting on the values $\mathbb{V}_\lambda$, see (2.16). The action of the conformal group on a function $f \in \mathcal{C}^\infty(\Omega, \mathbb{V}_\lambda)$ is given by

$$(g \cdot f)(x) = |cx+d|^{-2w} L\left(\frac{(cx+d)^\sim}{|cx+d|}\right) f((ax+b)(cx+d)^{-1}) \tag{3.11}$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(m)$; the conformal weight w is equal to $\frac{m-1}{2}$, as mentioned above. It can be proved (see [65]) that also $g \cdot f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$.

The following result, translated to our case of operators, then holds:

Theorem 18 ([65]). *If $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$ is a null solution of the conformally invariant differential operator $\mathcal{Q}_\lambda$, then so is the transformed function $g \cdot f$, given in (3.11), with g an element of the conformal group.*

This theorem will be essential in the construction of the fundamental solutions for our higher spin Dirac operator $\mathcal{Q}_{k,l}$ in chapter 7. We also use this result in chapter 10, when we define embedding factors for null solutions in the kernel space of the operator $\mathcal{Q}_{k,l}$.

3.3.2 Surjectivity

Based on a result from [15], it follows from Theorem 15 that the higher spin Dirac operator $\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$ is elliptic. A classical theorem of the theory of differential equations states that any linear elliptic differential operator $\mathcal{D}$ is surjective in smooth category (see e.g. [69]). In other words, the equation

$$\mathcal{D}f = g$$

has a solution for any $\mathcal{C}^\infty$ right-hand side. By means of this result, it was proved in [56] that

Proposition 9. *For any higher spin Dirac operator $\mathcal{Q}_\lambda$, the equation*

$$\mathcal{Q}_\lambda f = g$$

has a polynomial solution for every polynomial right-hand side.

This means that the higher spin Dirac operators, acting on polynomials in $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$, are surjective.

3.3.3 Twisted Dirac operator

To end this chapter, we introduce the concept of the *twisted Dirac operator*, acting on polynomials with values in $\mathbb{V}_\mu \otimes \mathbb{S}$, where $\mathbb{V}_\mu$ is a representation of $\mathrm{Spin}(m)$ with highest weight μ. Let $\{v_i\}$ be a basis of this vector space. The twisted Dirac operator is the operator $\mathcal{D}_T$ acting on the space of polynomials $\mathcal{P}(\mathbb{R}^m, \mathbb{V}_\mu \otimes \mathbb{S})$ with values in $\mathbb{V}_\mu \otimes \mathbb{S}$. This action is defined by the formula

$$\mathcal{D}_T : \sum_i v_i \otimes s_i(x) \to \sum_i v_i \otimes \partial_x s_i(x)$$

where $s_i(x)$ are $\mathbb{S}$-valued polynomials. Similarly to the Dirac operator, the twisted Dirac operator is a $\mathrm{Spin}(m)$-invariant first-order differential operator with constant coefficients. For more information, we refer to [61].

Chapter 4

Rarita-Schwinger operators

The Rarita-Schwinger operator is a typical example of a higher spin Dirac operator. In [18, 19] this operator is studied from the point of view of Clifford analysis and representation theory.

The study of higher spin Dirac operators, which originates from geometry and physics, is important. As mentioned previously, it follows from results by e.g. [65, 38, 15] (see also Theorem 15 in section 3.3) that there exists a sequence of similar elliptic conformally invariant first-order differential operators acting on functions with values in more complicated spinor-representations. The simplest operator of this sequence, apart from the standard Dirac operator, acts on functions with values in a $\mathrm{Spin}(m)$-module with highest weight $(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and is denoted by $\mathcal{R}_1$. The focus of this chapter lies on the higher spin Dirac operator $\mathcal{R}_k$ ($k \in \mathbb{N}$), which acts on functions with values in a representation of $\mathrm{Spin}(m)$ with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. This type of operators is called Rarita-Schwinger operators, inspired by a result in theoretical physics, where the Rarita-Schwinger equation is the relativistic field equation of spin-$\frac{3}{2}$ fermions. Another example of an elliptic conformally invariant first-order differential operator is given in the next chapter. In chapter 6, the notion of $\mathcal{R}_k$ is generalised to that of an operator $\mathcal{Q}_{k,l}$ that acts on $\mathrm{Spin}(m)$-modules of highest weight $(k + \frac{1}{2}, l + \frac{1}{2}, \ldots, \frac{1}{2})$ with $k \geq l$.

As mentioned in section 2.3.5, the vector space $\mathcal{M}_k$ of spherical monogenics of degree k is a realisation of a $\mathrm{Spin}(m)$-representation with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Hence, in Clifford analysis, the Rarita-Schwinger operator is defined as the elliptic conformally invariant first-order differential operator

$$\mathcal{R}_k : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k).$$

From the perspective of Clifford analysis, the operator $\mathcal{R}_k$ is interesting for several reasons. First, $\mathcal{R}_k$ acts on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$ and the space $\mathcal{M}_k$ plays a fundamental role in Clifford analysis. In particular, this vector space occurs in the refined Fischer decomposition (see Theorem 11) with respect to the vector variable u:

$$\mathcal{H}_k \otimes \mathbb{S} = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}$$

or, equivalently,

$$\mathcal{H}_k \otimes \mathbb{S} \ni H_k = P_k + uP_{k-1}$$

with every P_j monogenic in u. Second, the Rarita-Schwinger operator $\mathcal{R}_k$ is a so-called *monogenic operator* (see [63, 64]); there exists a similar decomposition, which translates to

$$\partial_x = \mathcal{R}_k + u\mathcal{T}_{k-1}^k$$

where $\mathcal{T}_{k-1}^k$ is the so-called dual twistor operator, which is the conformally invariant first-order differential operator

$$\mathcal{T}_{k-1}^k : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{k-1}).$$

For more information about monogenic operators, which transform polynomials of type $u^s P_k$ to polynomials of type $u^{s'} P_{k'}$, we refer to [63, 64].

In [18], the operator $\mathcal{R}_k$ is discussed mainly from a Clifford analysis point of view. One can also use abstract represention spaces only and study a function theory by means of geometrical and representation theoretical tools, which includes a general definition of equations on any Riemannian manifold with a given spin structure. We will not follow this approach, for which we refer the interested reader to [19].

In the next sections, an overview of several results of [18], which have been rearranged and adapted to the notations used this thesis, is given. Explicitly, we discuss the construction of $\mathcal{R}_k$, the description of its null solutions and the geometry of $\mathrm{Ker}_h\mathcal{R}_k$. In [18] some function theoretical results with respect to the operator $\mathcal{R}_k$ are obtained, such as the fundamental solution and basic integral formulae, but we have chosen to discuss these topics in more generality in chapter 7.

Although the case of the operator $\mathcal{Q}_{k,l}$ presents complications that were not yet discovered in the Rarita-Schwinger case, the importance of this chapter cannot be underestimated because the constructions, lemmas and propositions have been an inspiration for the results in the next chapters.

4.1 Construction of $\mathcal{R}_k$

The ‘monogenic decomposition’ of the Dirac operator, given by $\partial_x = \mathcal{D}_1 + \mathcal{D}_2$, suggests an expression for the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ in terms of ∂_x and the Gamma operator Γ_u. Even though this provides a definition for $\mathcal{R}_k$ and $\mathcal{T}^k_{k-1}$ (with $\mathcal{D}_1 = \mathcal{R}_k$ and $\mathcal{D}_2 = u\mathcal{T}^k_{k-1}$), we will find the explicit form for the latter operators using a different approach, which will be easier to generalise. To that end, we need the monogenic Fischer decomposition of harmonics once again:

$$\mathcal{H}_k \otimes \mathbb{S} = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}. \tag{4.1}$$

This decomposition gives rise to the operators in Figure 4.1 and Lemma 9.

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S}) \arrow[r, bend left, "\partial_x"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S}) \\
\| & \| \\
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \arrow[r, "\mathcal{R}_k"] \arrow[ddr, "u\mathcal{T}^k_{k-1}"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \\
\oplus & \oplus \\
\mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) & \mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1})
\end{tikzcd}$$

Figure 4.1: Introducing: operators $\mathcal{R}_k$, $\mathcal{T}^k_{k-1}$.

First, we prove that

Lemma 8. *The projection operators*

$$\pi_1 : \mathcal{H}_k \otimes \mathbb{S} \to \mathcal{M}_k \quad \text{and} \quad \pi_2 : \mathcal{H}_k \otimes \mathbb{S} \to \mathcal{M}_{k-1}$$

are given by

$$\pi_1 := \left(1 + \frac{u\partial_u}{m+2k-2}\right) \quad \text{and} \quad \pi_2 := -\frac{1}{m+2k-2}\partial_u,$$

respectively.

Proof. If $H_k \in \mathcal{H}_k \otimes \mathbb{S}$, $P_k \in \mathcal{M}_k$ and $P_{k-1} \in \mathcal{M}_{k-1}$, it follows from (4.1) that

$$H_k = P_k + uP_{k-1}.$$

An explicit expression for P_k and P_{k-1} is found as follows. Since

$$\partial_u H_k = -(m + 2\mathbb{E}_u)P_{k-1},$$

we have

$$P_{k-1} = -\frac{1}{m+2k-2}\partial_u H_k =: \pi_2 H_k.$$

This leads to an expression for P_k:

$$\begin{aligned} P_k &= H_k - uP_{k-1} \\ &= H_k + \frac{u\partial_u}{m+2k-2} H_k \\ &= \left(1 + \frac{u\partial_u}{m+2k-2}\right) H_k =: \pi_1 H_k \end{aligned}$$

which concludes the proof. □

Lemma 9. *Let $f(x;u)$ be a polynomial in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$.*
(i) The Rarita-Schwinger operator is the unique (up to a multiplicative constant) elliptic conformally invariant first-order differential operator

$$\mathcal{R}_k : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$$

defined as

$$\mathcal{R}_k f(x;u) := \pi_1 \partial_x f(x;u) = \left(1 + \frac{u\partial_u}{m+2k-2}\right) \partial_x f(x;u). \tag{4.2}$$

(ii) The dual twistor operator is the unique (up to a multiplicative constant) conformally invariant first-order differential operator

$$\mathcal{T}^k_{k-1} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{k-1})$$

defined as

$$\begin{aligned} \mathcal{T}^k_{k-1} f(x;u) &:= \pi_2 \partial_x f(x;u) \\ &= -\frac{1}{m+2k-2}\partial_u \partial_x f(x;u) = \frac{2}{m+2k-2}\langle \partial_u, \partial_x\rangle f(x;u). \end{aligned} \tag{4.3}$$

Proof. It follows from the previous lemma that

$$\begin{aligned} \pi_1 &: \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \\ \pi_2 &: \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{k-1}). \end{aligned}$$

Let $f(x;u)$ be a polynomial in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$. It is not difficult to see that $\partial_x f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S})$, whence

$$\mathcal{R}_k f(x;u) := \pi_1 \partial_x f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$$

and

$$\mathcal{T}_{k-1}^k f(x;u) := \pi_2 \partial_x f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{k-1}).$$

The explicit expressions of these operators follow from Lemma 8. $\square$

Remark 16. *The dual twistor operator $\mathcal{T}_{k-1}^k$ equals $\langle \partial_u, \partial_x \rangle$, up to a multiplicative constant. We will usually refer to $\langle \partial_u, \partial_x \rangle$ instead of $\mathcal{T}_{k-1}^k$ as the dual twistor operator.*

In Figure 4.2 the twistor operator $\mathcal{T}_k^{k-1}$ is introduced.

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S}) \arrow[r, bend left, "\partial_x"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S}) \\
\| & \| \\
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \\
\oplus & \oplus \\
\mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1}) \arrow[ruu, "\mathcal{T}_k^{k-1}u^{-1}"] & \mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{M}_{k-1})
\end{tikzcd}$$

Figure 4.2: Introducing: operator $\mathcal{T}_k^{k-1}$.

This is the unique (up to a multiplicative constant) conformally invariant first-order differential operator, defined as

$$\begin{aligned}
\mathcal{T}_k^{k-1} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_{k-1}) &\rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k) \\
f(x;u) &\mapsto \mathcal{T}_k^{k-1} f(x;u) := \pi_1 \partial_x u f(x;u).
\end{aligned} \tag{4.4}$$

4.2 Description of $\mathbf{Ker}_h\mathcal{R}_k$

In this section we study h-homogeneous polynomial null solutions of $\mathcal{R}_k$, i.e. polynomials $f(x;u)$ that satisfy $\mathcal{R}_k f(x;u) = \pi_1 \partial_x f(x;u) = 0$. This vector space of null solutions is denoted by $\mathrm{Ker}_h \mathcal{R}_k$. There are two ways for $f(x;u)$ to satisfy this condition: either $\partial_x f(x;u) = 0$, or $\partial_x f(x;u) \neq 0$ and $\pi_1 \partial_x f(x;u) = 0$. The first type is called 'solutions of type A' and the second type, 'solutions of type B'. In what follows, we put $h \geq k$.

4.2.1 Solutions of type A or the double monogenics

The solutions of type A are polynomials $f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$ that are homogeneous of degree h in x and that satisfy $\partial_x f(x;u) = 0$ and $\partial_u f(x;u) = 0$. This means that

$$f \in \mathcal{M}_{h,k}$$

and recall from (6.6) that $\mathcal{M}_{h,k}$ decomposes in $k+1$ Spin(m)-irreducible summands:

$$\mathcal{M}_{h,k} = \bigoplus_{j=0}^{k} \langle u, \partial_x \rangle^j \mathcal{S}_{h+j,k-j}.$$

4.2.2 Solutions of type B or the induction principle

This time, let $f(x;u)$ be a polynomial in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{M}_k)$, homogeneous of degree h in x with $\partial_x f(x;u) \neq 0$. If $f \in \mathrm{Ker}_h \mathcal{R}_k$, then the following equivalence holds:

$$\left(1 + \frac{u\partial_u}{m+2k-2}\right)\partial_x f = 0 \;\Leftrightarrow\; \partial_x f = \frac{2}{m+2k-2} u \langle \partial_u, \partial_x \rangle f. \tag{4.5}$$

The next lemma states an important result.

Lemma 10. *For all integers $h \geq k > 0$, one has*

$$\langle \partial_u, \partial_x \rangle : \mathrm{Ker}_h \mathcal{R}_k \to \mathrm{Ker}_{h-1} \mathcal{R}_{k-1},$$

i.e. the dual twistor operator maps solutions to solutions.

Proof. We act with $\langle \partial_u, \partial_x \rangle$ on (4.5):

$$\begin{aligned}
\partial_x \langle \partial_u, \partial_x \rangle f &= \langle \partial_u, \partial_x \rangle \partial_x f \\
&= \frac{2}{m+2k-2} [\langle \partial_u, \partial_x \rangle, u] \langle \partial_u, \partial_x \rangle f + \frac{2}{m+2k-2} u \langle \partial_u, \partial_x \rangle^2 f \\
&= \frac{2}{m+2k-2} \partial_x \langle \partial_u, \partial_x \rangle f + \frac{2}{m+2k-2} u \langle \partial_u, \partial_x \rangle^2 f
\end{aligned}$$

which can be written as

$$\partial_x \langle \partial_u, \partial_x \rangle f = \frac{2}{m+2k-4} u \langle \partial_u, \partial_x \rangle^2 f.$$

This is equivalent with

$$\mathcal{R}_{k-1}\langle \partial_u, \partial_x \rangle f = \left(1 + \frac{u\partial_u}{m+2k-4}\right) \partial_x \langle \partial_u, \partial_x \rangle f = 0$$

which concludes the proof. □

Using this lemma, the result in (4.5) implies that

$$f \in \mathrm{Ker}_h \mathcal{R}_k \ \Rightarrow \ \partial_x f = ug$$

with $g \in \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$. The following question arises: do we also have that

$$\left\{ \begin{array}{l} \partial_x f = ug \\ \partial_u f = 0 \end{array} \right. \Rightarrow f \in \mathrm{Ker}_h \mathcal{R}_k$$

for a certain $g \in \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$? The first equation always has a solution if we take $g = \mathcal{T}^k_{k-1} f$. It is not straightforward to find a solution $f(x;u)$ of this inhomogeneous equation which is monogenic in u. In [24] one studies a more general system of equations of polynomials which are homogeneous in the vector variables x, u:

$$\left\{ \begin{array}{l} \partial_x f = h_1 \\ \partial_u f = h_2 \end{array} \right. \tag{4.6}$$

Systems of this type with f, h_1 and h_2 in more general function spaces have been studied in e.g. [24, 59]. An alternative method, based on a result in [25], is also discussed in [18]. If f is a solution of (4.6), it is not difficult to see that h_1 and h_2 have to satisfy the following *compatibility conditions*:

$$\left\{ \begin{array}{l} \Delta_u \partial_x f = \Delta_u h_1 \\ -\partial_x \Delta_u f = \partial_x \partial_u h_2 \\ -\partial_u \Delta_x f = \partial_u \partial_x h_1 \\ \Delta_x \partial_u f = \Delta_x h_2 \end{array} \right.$$

which is equivalent to

$$\left\{ \begin{array}{l} \Delta_u h_1 + \partial_x \partial_u h_2 = 0 \\ \partial_u \partial_x h_1 + \Delta_x h_2 = 0 \end{array} \right.$$

In case $h_1 = ug$ and $h_2 = 0$, these conditions reduce to

$$\left\{ \begin{array}{l} \Delta_u (ug) = 0 \\ \partial_u \partial_x (ug) = 0 \end{array} \right. \tag{4.7}$$

The first condition is equivalent with the monogeneity of g in the variable u. It is a necessary condition, since

$$\Delta_u(ug) = 0 \;\Leftrightarrow\; (2 - u\partial_u)\partial_u g = 0.$$

It is also sufficient, which can be proved as follows. Using the monogenic Fischer decomposition, we write g as

$$g = g_{k-1} + ug_{k-2} + u^2 g_{k-3} + \cdots + u^{k-1} g_0$$

with $\mathbb{E}_u g_{k-i} = (k-i)g_{k-i}$ and $\partial_u g_{k-i} = 0$ for all $1 \leq i \leq k$. As ug is harmonic, we have from the monogenic Fischer decomposition of harmonics that

$$ug = f_1 + uf_2$$

with $\partial_u f_1 = \partial_u f_2 = 0$. Hence, it immediately follows that

$$g_{k-2} = \ldots = g_0 = 0$$

which implies that $\partial_u g = 0$. Using the fact that $g \in \mathcal{M}_{k-1}$, the second condition of (4.7) leads to

$$\begin{aligned}
&\partial_u \partial_x (ug) = 0\\
&\Leftrightarrow -2[\langle \partial_u, \partial_x \rangle, u]g - 2u\langle \partial_u, \partial_x \rangle g - \partial_x \{\partial_u, u\} g = 0\\
&\Leftrightarrow -2\partial_x g + (m + 2\mathbb{E}_u)\partial_x g - 2u\langle \partial_u, \partial_x \rangle g = 0\\
&\Leftrightarrow (m + 2\mathbb{E}_u - 2)\partial_x g - 2u\langle \partial_u, \partial_x \rangle g = 0\\
&\Leftrightarrow (m + 2\mathbb{E}_u - 2)\left(\mathbf{1} + (m + 2\mathbb{E}_u - 2)^{-1} u\partial_u\right)\partial_x g = 0\\
&\Leftrightarrow \mathcal{R}_{k-1} g = 0.
\end{aligned}$$

Type B solutions for $\mathcal{R}_k$ are thus equivalent with elements of $\mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$; their structure may be described through an inductive procedure.

Remark 17. *The compatibility conditions (4.7) signify that the kernel space for the operator $\mathcal{R}_{k-1}$ can be embedded into the kernel space for $\mathcal{R}_k$. The embedding goes by means of operators that will be defined in the next section.*

The previous results are combined into the following theorem.

Theorem 19 (The induction principle [18])**.** *For all integers $h \geq k$, one has*

$$\mathrm{Ker}_h \mathcal{R}_k = \mathcal{M}_{h,k} \oplus \partial_x^{-1}\left(u\mathrm{Ker}_{h-1}\mathcal{R}_{k-1}\right)$$

where ∂_x^{-1} associates to each $g \in \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$ the unique solution of the system

$$\begin{cases} \partial_x f = ug \\ \partial_u f = 0 \end{cases}$$

which is orthogonal to $\mathcal{M}_{h,k}$ with respect to the Fischer inner product.

Remark 18. *The null solutions of $\mathcal{R}_k$ are polymonogenic in x: if $f \in \mathrm{Ker}_h\mathcal{R}_k$, then f satisfies one of the conditions: $\partial_x f = 0$, $\partial_x^3 f = 0$, $\ldots$, $\partial_x^{2k+1} f = 0$.*

4.3 Decomposition of $\mathrm{Ker}_h\mathcal{R}_k$

4.3.1 Geometry of solutions

In this section, we investigate how the space $\mathrm{Ker}_h\mathcal{R}_k$ can be decomposed by recursion into Spin(m)-irreducible summands. For example, consider the case $k = 1$. Theorem 19 leads to

$$\begin{aligned} \mathrm{Ker}_h\mathcal{R}_1 &= \mathcal{M}_{h,1} \oplus \partial_x^{-1}\left(u\mathrm{Ker}_{h-1}\partial_x\right) \\ &= \mathcal{S}_{h,1} \oplus \langle u, \partial_x\rangle\mathcal{M}_{h+1} \oplus \partial_x^{-1}\left(u\mathcal{M}_{h-1}\right) \\ &= \mathcal{S}_{h,1} \oplus \langle u, \partial_x\rangle\mathcal{M}_{h+1} \oplus \mu\mathcal{M}_{h-1} \\ &\cong (h+\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}) \oplus (h+\tfrac{3}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}) \oplus (h-\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}). \end{aligned} \tag{4.8}$$

The three summands in (4.8) are visualised as a triangle in Figure 4.3.

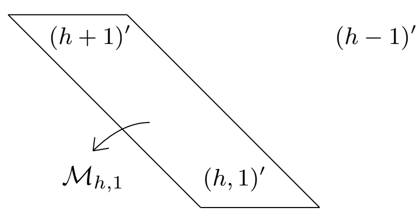


Figure 4.3: The decomposition of $\mathrm{Ker}_h\mathcal{R}_1$.

It it shown in [18] that the map

$$\mu : \mathcal{M}_{h-1} \to \mathrm{Ker}_h\mathcal{R}_1$$

defined as

$$\mu\mathcal{M}_{h-1} \cong \big(m\langle x,u\rangle + ux + |x|^2\langle u,\partial_x\rangle\big)\,\mathcal{M}_{h-1},$$

gives an explicit realisation of the representation $(h-\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ inside $\mathrm{Ker}_h\mathcal{R}_1$.

Applying Theorem 19 with $k=2$ leads to the decomposition

$$\begin{aligned}\mathrm{Ker}_h\mathcal{R}_2 &= \mathcal{M}_{h,2} \oplus \partial_x^{-1}\left(u\mathrm{Ker}_{h-1}\mathcal{R}_1\right)\\ &\cong (h,2)' \oplus (h+1,1)' \oplus (h+2)' \oplus (h-1,1)' \oplus (h)' \oplus (h-2).\end{aligned} \tag{4.9}$$

These summands are visualised in Figure 4.4.

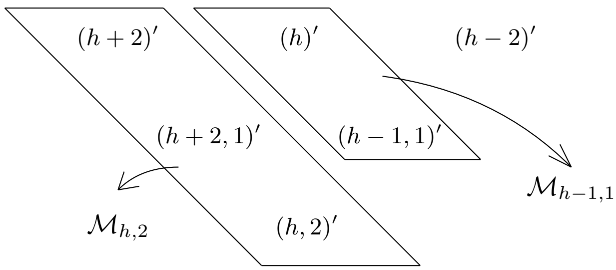


Figure 4.4: The decomposition of $\mathrm{Ker}_h\mathcal{R}_2$.

However, finding an explicit expression for

$$\partial_x^{-1}\left(u\mathrm{Ker}_{h-1}\mathcal{R}_{k-1}\right)$$

with $k>1$, or, equivalently, solving

$$\partial_x f = ug$$

with $g \in \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$, is much more complicated, due to the fact that the space $\mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$ consists of $\frac{k(k+1)}{2}$ irreducible pieces. This number can easily be found through induction on k as follows. We showed that $\mathrm{Ker}_h\mathcal{R}_1$ consists of 3 irreducible summands. By means of Theorem 19 it is not difficult to see that $\mathrm{Ker}_h\mathcal{R}_k$ decomposes into $k+1+\frac{k(k+1)}{2}=\frac{(k+1)(k+2)}{2}$ irreducible pieces. These summands, denoted by their highest weight, are represented in the form of a triangle in Figure 4.5.

$$
\begin{array}{c}
(h+k)' \,,\, (h+k-2)' \,,\, (h+k-2)' \,, \cdots \,,\, (h+k-2)' \,,\, (h-k+2)' \,,\, (h-k)' \\
(h+k-1,1)' \;,\; (h+k-3,1)' \;,\; \cdots \;,\; (h-k+3,1)' \;,\; (h-k+1,1)' \\
\ddots \qquad\qquad \vdots \qquad\qquad \iddots \\
(h+2,k-2)' \quad , \quad (h,k-2)' \quad , \quad (h-2,k-2)' \\
(h+1,k-1)' \quad , \quad (h-1,k-1)' \\
(h,k)'
\end{array}
$$

Figure 4.5: The decomposition of $\mathrm{Ker}_h\mathcal{R}_k$, also known as the “Christmas tree”.

We write the decomposition in Figure 4.5 as follows. In (4.9) it is shown that

$$\mathrm{Ker}_h\mathcal{R}_2 \cong \mathcal{M}_{h,2} \oplus \mathcal{M}_{h-1,1} \oplus \mathcal{M}_{h-2}$$

and the summands of the right-hand side constitute the ‘diagonal lines’ in the triangle in Figure 4.4. Similarly, we can write Theorem 19 as

$$\mathrm{Ker}_h\mathcal{R}_k \cong \mathcal{M}_{h,k} \oplus \mathcal{M}_{h-1,k-1} \oplus \cdots \oplus \mathcal{M}_{h-k+1,1} \oplus \mathcal{M}_{h-k}.$$

where $\mathcal{M}_{h,k}$ is the left edge of the triangle in Figure 4.5, and $\mathcal{M}_{h-k}$ is the vertex on the right. We end this section with a visualisation of Theorem 19.

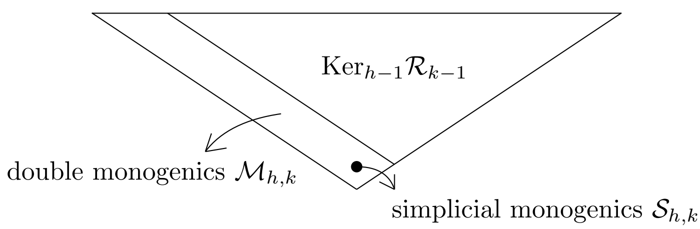


Figure 4.6: Geometry of $\mathrm{Ker}_h\mathcal{R}_k$ (Theorem 19).

4.3.2 Embedding factors of solutions

In this section, we solve the following problem:

$$\partial_x f = ug \;\Rightarrow\; f = ?$$

with $g \in \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$. To this end, we introduce an operator that is essential to construct embedding factors: the inversion operator I_R.

Definition 4. *Let $f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S})$. Denote by I_R be the inversion operator corresponding to the operator $\mathcal{R}_k$, defined as*

$$I_R f(x;u) = \frac{x}{|x|^m} f\left(\frac{x}{|x|^2}; \frac{xux}{|x|^2}\right).$$

Some properties are given in the next lemma.

Lemma 11. *One has*

(*i*) *I_R is Spin(m)-invariant*

(*ii*) $[I_R, \pi_1] = 0$

(*iii*) $(I_R)^2 = -\mathbf{1}$.

Proof. We refer to Lemma 32, where we prove a more general result. □

Because the higher spin operator $\mathcal{R}_k$ is conformally invariant, it follows from Theorem 18 that this inversion operator preserves solutions:

$$I_R : \mathrm{Ker}_h\mathcal{R}_k \to \mathrm{Ker}_{1-m-h}\mathcal{R}_k.$$

The Laplace operator Δ_x also preserves solutions:

$$\Delta_x : \mathrm{Ker}_h\mathcal{R}_k \to \mathrm{Ker}_{h-2}\mathcal{R}_k.$$

Combining these two results, we have

$$I_R\Delta_x I_R : \mathrm{Ker}_h\mathcal{R}_k \to \mathrm{Ker}_{h+2}\mathcal{R}_k.$$

Lemma 12. *The operator*

$$I_R\Delta_x I_R\Delta_x : \mathrm{Ker}_h\mathcal{R}_k \to \mathrm{Ker}_h\mathcal{R}_k$$

acts by scalar multiplication on each irreducible Spin(m)-module in $\mathrm{Ker}_h\mathcal{R}_k$.

Proof. This follows from Schur's lemma, see [60] or Lemma 2. □

This means that we can invert $\partial_x f = ug$ by the identity

$$f = a_{hk} I_R \Delta_x I_R \partial_x ug$$

for a suitable normalisation a_{hk}.

Remark 19. *In chapter 10, we formulate an alternative embedding factor for irreducible summands in* $\mathrm{Ker}_h\mathcal{R}_k$.

If we denote the operator $I_R\Delta_x I_R\partial_x u$ by Λ_u, Theorem 19 implies that

Theorem 20. *One has*

$$\mathrm{Ker}_h\mathcal{R}_k = \bigoplus_{i=0}^{k}(\Lambda_u)^i\mathcal{M}_{h-i,k-i} = \bigoplus_{i=0}^{k}\bigoplus_{j=0}^{k-i}(\Lambda_u)^i\langle u,\partial_x\rangle^j\mathcal{S}_{h-i+j,k-i-j}.$$

4.3.3 Useful results

Another useful operator is $I_R\partial_x I_R$; an explicit expression is given in the next proposition.

Proposition 10. *The operator $I_R\partial_x I_R$ is an endomorphism on the vector space $\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{H}_k\otimes\mathbb{S})$ and satisfies*

$$I_R\partial_x I_R = |x|^2\partial_x + 2\langle x,u\rangle\partial_u - 2u\langle x,\partial_u\rangle.$$

Proof. We refer to Proposition 26, where we prove a more general result. □

We end this section with a nice lemma.

Lemma 13. *For $f\in\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{M}_k)$, one has*

$$\pi_1 I_R\partial_x I_R f = |x|^2\mathcal{R}_k f.$$

Proof. It follows from Proposition 10 that

$$\pi_1 I_R\partial_x I_R f \;=\; \pi_1\left(|x|^2\partial_x + 2\langle x,u\rangle\partial_u - 2u\langle x,\partial_u\rangle\right)f.$$

Since $\partial_u f = 0$ and

$$\pi_1(u\langle x,\partial_u\rangle)f = \big(m+2\mathbb{E}_u + u\partial_u\big)(u\langle x,\partial_u\rangle)f = 0,$$

we have

$$\pi_1 I_R\partial_x I_R f \;=\; |x|^2\pi_1\partial_x f = |x|^2\mathcal{R}_k f.$$

□

Chapter 5

The operator $\mathcal{Q}_{1,1}$

In this chapter we focus on a special case of higher spin Dirac operators acting on functions with values in the space of spinor-valued differential forms, which are studied in a more general setting in [16, 17]. Working in Euclidean space, these higher spin Dirac operators can be thought of as elliptic conformally invariant first-order differential operators $\mathcal{Q}_{\lambda_k}$:

$$\mathcal{Q}_{\lambda_k} : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_{\lambda_k}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_{\lambda_k})$$

with $\mathbb{V}_{\lambda_k}$ an irreducible Spin(m)-module with highest weight

$$\lambda_k = (\underbrace{\frac{3}{2}, \ldots, \frac{3}{2}}_{k}, \frac{1}{2}, \ldots, \frac{1}{2}).$$

As mentioned in section 2.3.7, the vector space of simplicial monogenics, denoted by $\mathcal{S}_{\lambda_k}$, forms a model for such an irreducible representation in Clifford analysis. Hence, we write

$$\begin{aligned} \mathcal{Q}_{\lambda_k} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) &\to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \\ f(x; u_1, \ldots, u_k) &\mapsto \mathcal{Q}_{\lambda_k} f(x; u_1, \ldots, u_k). \end{aligned}$$

The aim of this chapter is to investigate the null solutions of the following elliptic conformally invariant first-order differential operator:

$$\begin{aligned} \mathcal{Q}_{1,1} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{1,1}) &\to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{1,1}) \\ f(x; u, v) &\mapsto \mathcal{Q}_{1,1} f(x; u, v) \end{aligned}$$

which is a special case of the operator $\mathcal{Q}_{\lambda_k}$ for $k = 2$. The decomposition of the kernel space of $\mathcal{Q}_{1,1}$ is essential in the construction of $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ (for all integers $k \geq l \geq 0$), which will be the topic of chapter 8.

Before moving on to the case $k = 2$, we construct the higher spin operator $\mathcal{Q}_{\lambda_k}$ for a general $k \leq n$. This construction requires more information about the decomposition of the space of spinor-valued forms, which is discussed in section 5.1. The outline of the construction of $\mathcal{Q}_{\lambda_k}$ is given in section 5.2. Furthermore, the h-homogeneous polynomial null solutions of $\mathcal{Q}_{\lambda_k}$ do not form an irreducible representation for the Spin group, similarly to the case of the Rarita-Schwinger operators $\mathcal{R}_k$. Using the notation (2.43), we have the following result from [16]:

Proposition 11. *Let k be an integer satisfying $1 < k \leq n$. Then*

$$\mathrm{Ker}_h \mathcal{Q}_{\lambda_k} \cong \mathcal{S}_{\lambda_{h,k}} \oplus \mathcal{S}_{\lambda_{h+1,k-1}} \oplus \mathcal{S}_{\lambda_{h-1,k-1}} \oplus \mathcal{S}_{\lambda_{h,k-2}}.$$

In section 5.3, we prove this proposition for the case $k = 2$ using Clifford analysis.

We start with some remarks about notations. Recall that the Dirac operator with respect to the Clifford vectors u_i is denoted by ∂_i instead of ∂_{u_i} (with $1 \leq i \leq k$). Furthermore, we adopt another short notation for the highest weight of an irreducible representation of $\mathrm{Spin}(m)$, in addition to λ_k and $\lambda_{h,k}$ introduced in (2.42) and (2.43), respectively. Consider the highest weight

$$\mu_k := (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0),$$

with respect to the standard representation of $\mathrm{Spin}(m)$. Note that μ_0 is the highest weight of $\mathbb{C}$. The vector space of simplicial harmonics that corresponds to the highest weight μ_k, is denoted by $\mathcal{H}_{\mu_k}$. In particular, we have $\mathcal{H}_{\mu_2} = \mathcal{H}_{1,1}$, $\mathcal{S}_{\lambda_2} = \mathcal{S}_{1,1}$ and $\mathcal{S}_{\lambda_{h,2}} = \mathcal{S}_{h,1,1}$.

5.1 Decomposition of spinor-valued forms

Discussing the decomposition of spinor-valued forms begins with the following proposition, which reveals how $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$ decomposes into $\mathrm{Spin}(m)$-irreducible summands. The proof (by means of Klimyk's formula) is similar to that of Proposition 16 in chapter 6 and will be postponed; therefore we present only the result here.

Proposition 12. *For all integers $0 \leq k \leq n$ one has*

$$\mathcal{H}_{\mu_k} \otimes \mathbb{S} \cong \mathbb{S} \oplus \mathcal{S}_{\lambda_1} \oplus \cdots \oplus \mathcal{S}_{\lambda_k}.$$

The next question is how to embed the vector spaces $\mathbb{S}, \mathcal{S}_{\lambda_1}, \ldots, \mathcal{S}_{\lambda_k}$ in the tensor product $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$. As mentioned in section 2.3.6, every element in $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$ can be written as a spinor-valued polynomial depending on the wedge product $u_1 \wedge u_2 \wedge \ldots \wedge u_k$ only. By means of this result, we show that the embedding maps arise in a natural way; they are defined in the diagram in Figure 5.1.

$$\begin{tikzcd}
\mathbb{S} & \mathcal{H}_{\mu_1} \otimes \mathbb{S} & \cdots & \mathcal{H}_{\mu_{k-1}} \otimes \mathbb{S} & \mathcal{H}_{\mu_k} \otimes \mathbb{S} \\
\parallel & \parallel & & \parallel & \parallel \\
\mathbb{S} \arrow[r, bend left, "X_1"] & X_1\,\mathbb{S} \arrow[l, bend left, "Y_1"] & \cdots & X_{k-1}\cdots X_2 X_1\,\mathbb{S} & X_k X_{k-1}\cdots X_2 X_1\,\mathbb{S} \\
& \oplus & & \oplus & \oplus \\
& \mathcal{S}_{\lambda_1} \arrow[loop below, "H_1"] & \cdots & X_{k-1}\cdots X_2\,\mathcal{S}_{\lambda_1} & X_k X_{k-1}\cdots X_2\,\mathcal{S}_{\lambda_1} \\
& & & \oplus & \oplus \\
& & \ddots & \vdots & \vdots \\
& & & \oplus & \oplus \\
& & & X_{k-1}\,\mathcal{S}_{\lambda_{k-2}} & X_k X_{k-1}\,\mathcal{S}_{\lambda_{k-2}} \\
& & & \oplus & \oplus \\
& & & \mathcal{S}_{\lambda_{k-1}} \arrow[loop below, "H_{k-1}"] \arrow[r, bend left, "X_k"] & X_k\,\mathcal{S}_{\lambda_{k-1}} \arrow[l, bend left, "Y_k"] \\
& & & & \oplus \\
& & & & \mathcal{S}_{\lambda_k} \arrow[loop below, "H_k"]
\end{tikzcd}$$

Figure 5.1: Construction of embedding maps.

In order to find an expression for the operators X_k and Y_k, consider the first two columns. The Fischer decomposition (3.6) implies that $X_1 = u_1$ and $Y_1 = \partial_1$.

The remaining operators X_k and Y_k for $k > 1$ can be written as

$$X_k = u_k - \sum_{j=1}^{k-1} u_j \langle u_k, \partial_j \rangle \;:\; \mathcal{H}_{\mu_{k-1}} \otimes \mathbb{S} \;\to\; \mathcal{H}_{\mu_k} \otimes \mathbb{S} \tag{5.1}$$

$$Y_k = \partial_k \;:\; \mathcal{H}_{\mu_k} \otimes \mathbb{S} \;\to\; \mathcal{H}_{\mu_{k-1}} \otimes \mathbb{S}. \tag{5.2}$$

The action of each of the operators X_k results in adding the variable u_k in such a way that the result is an element of $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$; the operator Y_k removes the variable u_k. Explicitly,

Lemma 14. *For every integer k and all $\psi \in \mathbb{S}$, one has*

$$\begin{aligned} X_k\big(X_{k-1} \cdots X_2 X_1\, \psi\big) &= (-1)^{\frac{k(k-1)}{2}}\, k!\, u_1 \wedge u_2 \wedge \cdots \wedge u_{k-1} \wedge u_k\, \psi \\ Y_k\big(X_k X_{k-1} \cdots X_2 X_1\, \psi\big) &= (-1)^{\frac{k(k+1)}{2}}\, k!\, (m-k+1)\, u_1 \wedge u_2 \wedge \cdots \wedge u_{k-1}\, \psi. \end{aligned}$$

Proof. The proof of the first identity goes as follows. By consecutively applying (5.1) on $\psi \in \mathbb{S}$, we obtain the sum

$$(-1)^{\frac{k(k-1)}{2}} \sum_{\sigma \in \mathfrak{S}_k} sgn(\sigma)\, u_{\sigma(1)} \cdots u_{\sigma(k)}$$

which equals

$$(-1)^{\frac{k(k-1)}{2}} k! u_1 \wedge u_2 \wedge \cdots \wedge u_{k-1} \wedge u_k,$$

by definition. The second result is a special case of the next useful identity. □

Lemma 15. *For every integer k one has*

$$\partial_k \underbrace{X_k X_{k-1} \cdots X_{k-l+1}}_{l \text{ factors}} \mathcal{S}_{\lambda_{k-l}} \;=\; l(-m+2k-l-1) X_{k-1} \cdots X_{k-l+1} \mathcal{S}_{\lambda_{k-l}}.$$

Proof. The proof goes by induction on the number of embedding factors, denoted by l. In case $l = 1$, we immediately have

$$\partial_k X_k \mathcal{S}_{\lambda_{k-1}} = (-m + 2(k-1)) \mathcal{S}_{\lambda_{k-1}}.$$

Making use of the results

$$\partial_k X_k = -m + 2(k-1) + \sum_{j=1}^{k-1} u_j \partial_j \tag{5.3}$$

$$\{\partial_j, X_k\} = (m - 2(k-1)) \langle u_k, \partial_j \rangle \quad (j < k) \tag{5.4}$$

where both sides act on $\mathcal{H}_{\mu_{k-1}} \otimes \mathbb{S}$, and

$$[X_i, \langle u_r, \partial_j \rangle] = 0 \qquad (j \le r < i), \tag{5.5}$$

we find, for $l = 2$, that

$$\partial_k X_k X_{k-1} \mathcal{S}_{k-2} \;=\; 2(-m + 2k - 3) X_{k-1} \mathcal{S}_{k-2}.$$

Assume now that the statement holds for $l-1$ factors. We introduce the following short notations:

$$\begin{aligned}
B_k^l &:= l(-m + 2k - 1 - l) \\
C_k &:= -m + 2k \\
\overline{X_j} &:= X_j X_{j-1} \dots X_{k-l+1} \mathcal{S}_{\lambda_{k-l}} \\
\widehat{X_j} &:= X_k X_{k-1} \dots X_{j+1} X_{j-1} \dots X_{k-l+1} \mathcal{S}_{\lambda_{k-l}}.
\end{aligned}$$

Note that $B_k^l = C_{k-1} + B_{k-1}^{l-1}$. By means of the induction hypothesis, we have

$$\begin{aligned}
\partial_k X_k \overline{X_{k-1}} = &\{\partial_k, X_k\} \overline{X_{k-1}} \\
= &C_{k-1} \overline{X_{k-1}} + B_{k-1}^{l-1} u_{k-1} \overline{X_{k-2}} + \sum_{j=1}^{k-2} u_j \{\partial_j, X_{k-1}\} \overline{X_{k-2}} \\
&- \sum_{j=1}^{k-2} u_j X_{k-1} \partial_j X_{k-2} \overline{X_{k-3}} \\
= &\left(C_{k-1} + C_{k-2}\right) \overline{X_{k-1}} - C_{k-3} X_{k-2} \widehat{X_{k-2}} \\
&+ \left(B_{k-1}^{l-1} - C_{k-2}\right) u_{k-1} \widehat{X_{k-1}} - \left(B_{k-2}^{l-2} - C_{k-3}\right) u_{k-2} \widehat{X_{k-3}} \\
&- \sum_{j=1}^{k-3} u_j X_{k-1} \partial_j X_{k-2} \overline{X_{k-3}},
\end{aligned}$$

which can be written as

$$\begin{aligned}
\partial_k \overline{X_k} = &\left(C_{k-1} + C_{k-2}\right) \overline{X_{k-1}} - C_{k-3} X_{k-2} \widehat{X_{k-2}} \\
&+ \dots + (-1)^{l-1} C_i X_{k-l+2} \widehat{X_{k-l+2}} - (-1)^{l-1} C_{k-l} X_{k-l+1} \widehat{X_{k-l+1}} \\
&+ \left(B_{k-1}^{l-1} - C_{k-2}\right) u_{k-1} \widehat{X_{k-1}} - \left(B_{k-2}^{l-2} - C_{k-3}\right) u_{k-2} \widehat{X_{k-2}} \\
&+ \dots + (-1)^{l-1} \left(B_{k-l+2}^{2} - C_{k-l+1}\right) u_{k-l+2} \widehat{X_{k-l+2}}.
\end{aligned}$$

Using $B_k^l = C_{k-1} + B_{k-1}^{l-1}$, this leads to

$$
\begin{aligned}
&\partial_k \overline{X_k} = B_k^l \overline{X_{k-1}} - B_{k-2}^{l-2} \left(X_{k-1}\widehat{X_{k-1}} + X_{k-2}\widehat{X_{k-2}} - u_{k-1}\widehat{X_{k-1}} \right) \\
&+ B_{k-3}^{l-3} \left(X_{k-2}\widehat{X_{k-2}} + X_{k-3}\widehat{X_{k-3}} - u_{k-2}\widehat{X_{k-2}} \right) \\
&+ \cdots + (-1)^{l-1} B_{k-l+2}^{2} \left(X_{k-l+3}\widehat{X_{k-l+3}} + X_{k-l+2}\widehat{X_{k-l+2}} - u_{k-l+3}\widehat{X_{k-l+3}} \right) \\
&- (-1)^{l-1} B_{k-l+1}^{1} \left(X_{k-l+2}\widehat{X_{k-l+2}} + X_{k-l+1}\widehat{X_{k-l+1}} - u_{k-l+2}\widehat{X_{k-l+2}} \right).
\end{aligned}
$$

The statement then follows from the observation that one has, for every $r \in \{k-l+1, \ldots, k-2\}$,

$$
X_{r+1}\widehat{X_{r+1}} + X_r\widehat{X_r} - u_{r+1}\widehat{X_{r+1}} = 0.
$$

Indeed, the left-hand side equals

$$
u_r \left(\langle u_{r+1}, \partial_r \rangle \widehat{X_{r+1}} + \widehat{X_r} \right) + \sum_{j=1}^{r-1} u_j \left(\langle u_{r+1}, \partial_j \rangle \widehat{X_{r+1}} + \langle u_r, \partial_j \rangle \widehat{X_r} \right) = 0,
$$

due to the anti-symmetry in u_r and u_{r+1}. This concludes the proof. $\square$

Proposition 12 can now be written as follows.

Corollary 1. *For all integers $0 \leq k \leq n$ one has*

$$
\mathcal{H}_{\mu_k} \otimes \mathbb{S} = \left(\bigoplus_{i=0}^{k-1} X_k X_{k-1} \cdots X_{i+1} \mathcal{S}_{\lambda_i} \right) \oplus \mathcal{S}_{\lambda_k}.
$$

Next, define the operator H_k on $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$ as the following combination of the two operators described above:

$$
H_k = X_k Y_k - Y_{k+1} X_{k+1} : \mathcal{H}_{\mu_k} \otimes \mathbb{S} \rightarrow \mathcal{H}_{\mu_k} \otimes \mathbb{S}. \tag{5.6}
$$

Acting on a summand $X_k \cdots X_i \mathcal{S}_{\lambda_{i-1}}$ in $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$, this can be visualised as

$$
\begin{tikzcd}
X_{k-1} \cdots X_i \, \mathcal{S}_{\lambda_{i-1}} \arrow[r, bend left, "X_k"] & X_k X_{k-1} \cdots X_i \, \mathcal{S}_{\lambda_{i-1}} \arrow[l, bend left, "Y_k"] \arrow[loop below, "H_k"] \arrow[r, bend left, "X_{k+1}"] & X_{k+1} X_k \cdots X_i \, \mathcal{S}_{\lambda_{i-1}} \arrow[l, bend left, "Y_{k+1}"]
\end{tikzcd}
$$

Using (5.1), (5.2) and Lemma 15, it can be shown that H_k acts on $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$ as

$$H_k = (m - 2k)\,\mathbf{1}.$$

Furthermore, we have the following relations for the operators, acting on $\mathcal{H}_{\mu_{k-1}} \otimes \mathbb{S}$ and $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$, respectively:

$$H_k X_k - X_k H_{k-1} = -2X_k, \quad H_{k-1} Y_k - Y_k H_k = 2Y_k.$$

If we ignore the subscripts, the operators $\{H_k, Y_k, X_k\}$ form a representation of $\mathfrak{sl}(2,\mathbb{C})$. In order to convert this set, subscripts included, into a proper $\mathfrak{sl}(2,\mathbb{C})$-representation, we introduce new operators. We use the operators X_k (resp. Y_k and H_k) to create a global operator X (resp. Y and H) that acts on the space $\mathcal{H} \otimes \mathbb{S}$, defined by

$$\mathcal{H} \otimes \mathbb{S} \cong \bigoplus_{k=0}^{m} (\mathcal{H}_{\mu_k} \otimes \mathbb{S}).$$

This decomposition represents the complete triangle that contains all $\mathbb{S}$-valued forms, which is visualised in Figure 5.2. In the even-dimensional case $m = 2n$, there is only one central column.

$$\begin{array}{cccccccc}
\mathbb{C} & \mathcal{H}_{\mu_1} & & \mathcal{H}_{\mu_n} & \mathcal{H}_{\mu_{n+1}} & & \mathcal{H}_{\mu_{m-1}} & \mathcal{H}_{\mu_m} \\
\otimes & \otimes & \cdots & \otimes & \otimes & \cdots & \otimes & \otimes \\
\mathbb{S} & \mathbb{S} & & \mathbb{S} & \mathbb{S} & & \mathbb{S} & \mathbb{S} \\
\wr\wr & \wr\wr & & \wr\wr & \wr\wr & & \wr\wr & \wr\wr \\
\mathbb{S} & \mathbb{S} & \cdots & \mathbb{S} & \mathbb{S} & \cdots & \mathbb{S} & \mathbb{S} \\
 & \oplus & & \oplus & \oplus & & \oplus & \\
 & \mathcal{S}_{\lambda_1} & \cdots & \mathcal{S}_{\lambda_1} & \mathcal{S}_{\lambda_1} & \cdots & \mathcal{S}_{\lambda_1} & \\
 & & & \oplus & \oplus & & & \\
 & & & \vdots & \vdots & & & \\
 & & & \oplus & \oplus & & & \\
 & & & \mathcal{S}_{\lambda_n} & \mathcal{S}_{\lambda_n} & & &
\end{array}$$

Figure 5.2: Triangle of spinor-valued forms in the odd-dimensional case.

The $(k+1)$th column in Figure 5.2 is the decomposition of $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$ $(0 \leq k \leq m)$. When acting on a specific column in the triangle, the operators X, Y and H should automatically adopt the corresponding index. This can be done using the operator $\mathbb{E} := \sum_{i=1}^{m} \mathbb{E}_{u_i}$. Acting with the operator $\mathbb{E}$ on the $(k+1)$th column yields multiplication with k. Furthermore, each row in Figure 5.2 represents a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. The action of this Lie algebra on the $(i+1)$th row looks as follows:

$$
\begin{tikzcd}
\mathcal{S}_{\lambda_i} \arrow[r, shift left, "X"] \arrow[loop below, "-m+2i"] & X_{i+1}\mathcal{S}_{\lambda_i} \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] \arrow[loop below, "-m+2(i+1)"] & \cdots \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] & X_{m-i+1} \cdots X_{i+1}\mathcal{S}_{\lambda_i} \arrow[l, shift left, "Y"] \arrow[loop below, "m-2i"]
\end{tikzcd}
$$

It follows that any nonzero element from the leftmost vector space $\mathcal{S}_{\lambda_i}$ can be seen as a lowest weight vector for an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module of dimension $m - 2i + 1$. The Casimir operator $\mathcal{C}$ acts on this representation by multiplication with $(m-2i)(m-2i+2)$. These considerations give rise to the following projection operators, which can be also found in [61]:

Proposition 13. *The Spin(m)-invariant projection operator from $\mathcal{H} \otimes \mathbb{S}$ onto the $(k+1)$th column $(0 \leq k \leq m)$ is defined as*

$$\Pi^{\bullet,k} := \prod_{\substack{j=0 \\ j \neq k}}^{m} \frac{\mathbb{E} - j}{k - j}.$$

There exists a Spin(m)-invariant projection from $\mathcal{H} \otimes \mathbb{S}$ onto the $(i+1)$th row $(0 \leq i \leq n)$, defined as

$$\Pi^{i,\bullet} := \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{\mathcal{C} - (m-2j)(m-2j+2)}{4(i+j-m-1)(i-j)}.$$

Combining these operators, the operator

$$\Pi^{i,k} := \Pi^{i,\bullet} \circ \Pi^{\bullet,k} = \Pi^{\bullet,k} \circ \Pi^{i,\bullet},$$

is the Spin(m)-invariant projection operator from $\mathcal{H} \otimes \mathbb{S}$ onto the element on the $(i+1)$th row and $(k+1)$th column.

Using this proposition, we construct three operators acting on $\mathcal{H} \otimes \mathbb{S}$:

$$\pi(X) \;=\; \sqrt{-1}\sum_{l=1}^{m} X_l\,\Pi^{\bullet,l-1},\;\; \pi(Y) \;=\; \sqrt{-1}\sum_{l=1}^{m} Y_l\,\Pi^{\bullet,l},\;\; \pi(H) \;=\; -\sum_{l=1}^{m} H_l\,\Pi^{\bullet,l}.$$

They form a representation π of $\mathfrak{sl}(2,\mathbb{C})$.

An alternative expression for the operator $\Pi^{i,\bullet}$ in Proposition 13 can be found by means of the decomposition in Corollary 1. If $\psi \in \mathcal{H}_{\mu_k} \otimes \mathbb{S}$, then there exist simplicial monogenics $\psi_i \in \mathcal{S}_{\lambda_i}$ $(0 \leq i \leq k)$ such that

$$\psi = \psi_k + \sum_{i=0}^{k-1} X_k X_{k-1} \cdots X_{i+1} \psi_i. \tag{5.7}$$

Denoting the projection operators on $\mathcal{S}_{\lambda_i}$ $(0 \leq i \leq k-1)$ by

$$\begin{aligned} \pi_i : \mathcal{H}_{\mu_k} \otimes \mathbb{S} \;&\rightarrow\; \mathcal{S}_{\lambda_i} \\ \psi \;&\mapsto\; \psi_i, \end{aligned}$$

the projection operator on $\mathcal{S}_{\lambda_k}$ is immediately found to be

$$\pi_k \;:=\; \mathbf{1} - X_k \pi_{k-1} - X_k X_{k-1} \pi_{k-2} - \;\cdots\; - X_k \cdots X_2 \pi_1 - X_k \cdots X_1 \pi_0.$$

An explicit expression for the remaining projection operators is obtained by acting with $\partial_{i+1} \cdots \partial_{k-1} \partial_k$ on (5.7). Invoking Lemma 15 leads to

$$\pi_i(\psi) \;=\; C_{k,i} \left(\mathbf{1} + \sum_{l=1}^{i} \frac{X_i \cdots X_{i-l+1} \partial_{i-l+1} \cdots \partial_i}{l! \prod_{j=0}^{l-1} (m - 2i + 2 + j)} \right) \partial_{i+1} \cdots \partial_{k-1} \partial_k \psi \tag{5.8}$$

with $C_{k,i}^{-1} = (k-i)! \prod_{j=0}^{k-i-1} (-m+k-1+i-j)$ and $0 \leq i \leq k-1$. In particular,

$$\pi_{k-1}(\psi) \;=\; \left(\mathbf{1} + \sum_{l=1}^{k-1} \frac{X_{k-1} \cdots X_{k-l} \partial_{k-l} \cdots \partial_{k-1}}{l! \prod_{j=0}^{l-1} (m - 2k + 4 + j)} \right) \frac{\partial_k \psi}{-m + 2k - 2}. \tag{5.9}$$

As a result, we can make the following identification between these operators and the operator from Proposition 13:

$$\Pi^{i,k} \;=\; X_k \cdots X_{i+1} \pi_i,$$

both considered as operators acting on $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$.

5.2 Construction of the operator $\mathcal{Q}_{\lambda_k}$

Because the construction of the elliptic conformally invariant first-order differential operator

$$\mathcal{Q}_{\lambda_k} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k})$$

is very similar to the construction of the operator $\mathcal{Q}_{k,l}$, which is investigated in chapter 6, we present only the outline in this section.

Translated to this case of operators, the twisted Dirac operator, denoted again by ∂_x, acts as an endomorphism on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S})$. The following implications then hold:

$$f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \Rightarrow f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S}) \Rightarrow \partial_x f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S}).$$

Due to the relation $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S}) \cong \mathcal{C}^\infty(\mathbb{R}^m) \otimes (\mathcal{H}_{\mu_k} \otimes \mathbb{S})$, projection on the summand $\mathcal{S}_{\lambda_k}$ in the decomposition of $\mathcal{H}_{\mu_k} \otimes \mathbb{S}$ in Proposition 12 defines the operator $\mathcal{Q}_{\lambda_k}$:

$$f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \Rightarrow \mathcal{Q}_{\lambda_k} f := \pi_k(\partial_x) f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}).$$

The action of the twisted Dirac operator gives rise to two operators defined in Figure 5.3.

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S}) \arrow[r, bend left, "\partial_x"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S}) \\
\cup & \cup \\
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \arrow[r, "\mathcal{Q}_{\lambda_k}"] \arrow[ddr] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \\
\oplus & \oplus \\
\mathcal{C}^\infty(\mathbb{R}^m, X_k\, \mathcal{S}_{\lambda_{k-1}}) \arrow[uur, dashed, "X_k \mathcal{T}_{\lambda_{k-1}}^{\lambda_k}"'] & \mathcal{C}^\infty(\mathbb{R}^m, X_k\, \mathcal{S}_{\lambda_{k-1}})
\end{tikzcd}$$

Figure 5.3: Introducing: operators $\mathcal{Q}_{\lambda_k}$ and $\mathcal{T}_{\lambda_{k-1}}^{\lambda_k}$.

There does not exist a conformally invariant operator from $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k})$ to the other $k-1$ summands in the decomposition of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_k} \otimes \mathbb{S})$. This follows from results in [38, 68] and was explained in Theorem 16.

This can also be verified using the language of Clifford analysis.

Lemma 16. *For $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k})$, one has $\partial_i \partial_j \partial_x f = 0$ for all $1 \leq i, j \leq k$.*

Proof. Applying the definition of the Euclidean inner product on $\partial_j \partial_x$, we find

$$\partial_i \partial_j \partial_x f = -2\partial_i \langle \partial_j, \partial_x \rangle f - \partial_i \partial_x \partial_j f = 0$$

which concludes the proof. □

Applying this lemma on the explicit expression of the projection operators (5.8), the projection of $\partial_x f$ on $\mathcal{S}_{\lambda_i}$ $(0 \leq i \leq k-2)$ leads to

$$\pi_0(\partial_x f) \ = \ \pi_1(\partial_x f) \ = \ \ldots \ = \ \pi_{k-2}(\partial_x f) \ = \ 0.$$

This means that we have for every $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k})$:

$$\partial_x f = f_k + X_k f_{k-1}$$

where $f_i := \pi_i(f) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_i})$ with $i \in \{k-1, k\}$. It follows from (5.9) that

$$f_{k-1} \ = \ \pi_{k-1}(\partial_x f) \ = \ -\frac{1}{m-2k+2} \partial_k \partial_x f$$

which immediately leads to

$$\begin{aligned} f_k \ &= \ \partial_x f - X_k \pi_{k-1}(\partial_x f) \\ &= \ \partial_x f + \frac{1}{m-2k+2} \left(u_k - \sum_{i=1}^{k-1} u_i \langle u_k, \partial_i \rangle \right) \partial_k \partial_x f. \end{aligned}$$

Definition 5. *For all integers $k > 1$ there exists a unique (up to a multiplicative constant) elliptic conformally invariant first-order differential operator $\mathcal{Q}_{\lambda_k}$ defined as*

$$\mathcal{Q}_{\lambda_k} \ : \ \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \ \to \ \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k})\,;\ f \mapsto \pi_k(\partial_x f).$$

This operator is explicitly given by

$$\mathcal{Q}_{\lambda_k} f = \left(\mathbf{1} + \frac{1}{m-2k+2} \left(u_k - \sum_{i=1}^{k-1} u_i \langle u_k, \partial_i \rangle \right) \partial_k \right) \partial_x f.$$

Remark 20. *In the special case that $k = 1$ and $u_1 = u$, we obtain the Rarita-Schwinger operator $\mathcal{R}_1$:*

$$\mathcal{Q}_{\lambda_1} = \mathcal{R}_1 = \left(\mathbf{1} + \frac{u\partial_u}{m}\right)\partial_x.$$

Definition 6. *For all integers $k > 1$ there exists a unique (up to a multiplicative constant) conformally invariant differential operator, called the dual twistor operator $\mathcal{T}^{\lambda_k}_{\lambda_{k-1}}$, defined as*

$$\mathcal{T}^{\lambda_k}_{\lambda_{k-1}} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_{k-1}})\,;\, f \mapsto \pi_{k-1}(\partial_x f).$$

This operator is explicitly given by

$$\mathcal{T}^{\lambda_k}_{\lambda_{k-1}} f \;=\; \frac{2}{m-2k+2}\langle \partial_k, \partial_x\rangle f.$$

5.3 Homogeneous null solutions of $\mathcal{Q}_{\lambda_k}$

We begin this section with the following result:

Lemma 17. *For $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda_k})$, one has that $\langle \partial_i, \partial_x\rangle\langle \partial_j, \partial_x\rangle f = 0$ for all $1 \leq i, j \leq k$.*

Proof. Because f is 1-homogeneous in every u_i $(1 \leq i \leq k)$, the statement is obviously true if $i = j$. In case $i \neq j$, it follows from the symmetry in i and j of the operator $\langle \partial_i, \partial_x\rangle\langle \partial_j, \partial_x\rangle$ and the anti-symmetry in i and j in the values of f that $\langle \partial_i, \partial_x\rangle\langle \partial_j, \partial_x\rangle f = 0$ for all $1 \leq i, j \leq k$. □

Next, two types of homogeneous null solutions $f(x; u_1, \ldots, u_k)$ for $\mathcal{Q}_{\lambda_k}$ are defined. We make a distinction between solutions of type A, defined as $\partial_x f = 0$, and solutions of type B, defined as $\partial_x f \neq 0$ and $\pi_k(\partial_x f) = 0$. Over the next two sections, we will look for obvious candidates for these solutions.

5.3.1 Solutions of type A

Note that the solutions of type A are equivalent with polynomials in the vector space $\mathcal{M}^s_{\lambda_{h,k}}$ defined in section 2.4.4:

$$\mathcal{M}^s_{\lambda_{h,k}} = \{f \in \mathcal{M}_{\lambda_{h,k}} \mid \langle u_i, \partial_j\rangle f = 0 \;\text{ for }\; 1 \leq i \neq j \leq k\}$$

which decomposes as

$$\mathcal{M}^s_{\lambda_{h,k}} = \mathcal{S}_{\lambda_{h,k}} \oplus \left(\langle u_k, \partial_x \rangle - \sum_{i=1}^{k-1} \langle u_i, \partial_x \rangle \langle u_k, \partial_i \rangle \right) \mathcal{S}_{\lambda_{h+1,k-1}}.$$

5.3.2 Solutions of type B

In order to describe the type B-solutions, we start with the following result:

Lemma 18. *For $f \in \mathrm{Ker}_h \mathcal{Q}_{\lambda_k}$, one has $\langle \partial_k, \partial_x \rangle f \in \mathcal{M}^s_{\lambda_{h-1,k-1}}$.*

Proof. As a consequence of Lemma 17, we have that

$$\begin{aligned} \mathcal{Q}_{\lambda_k} f = 0 \;\Rightarrow\; & \langle \partial_k, \partial_x \rangle \left[\partial_x + \frac{2}{m-2k+2} \left(u_k - \sum_{i=1}^{k-1} u_i \langle u_k, \partial_i \rangle \right) \partial_k \partial_x \right] f = 0 \\ \Leftrightarrow\; & \partial_x \langle \partial_k, \partial_x \rangle f = 0. \end{aligned}$$

Furthermore, it is easily verified that $\langle \partial_k, \partial_x \rangle f$ is an element of $\mathcal{S}_{\lambda_{k-1}}$. □

Since

$$\begin{aligned} \mathcal{Q}_{\lambda_k} f = 0 \;\Leftrightarrow\; \partial_x f &= \frac{2}{m-2k+2} \left(u_k - \sum_{i=1}^{k-1} u_i \langle u_k, \partial_i \rangle \right) \langle \partial_k, \partial_x \rangle f \\ \Leftrightarrow\; \partial_x f &= \frac{2}{m-2k+2} X_k \underbrace{\langle \partial_k, \partial_x \rangle f}_{\substack{\Cap \\ \mathcal{M}^s_{\lambda_{h-1,k-1}}}}, \end{aligned}$$

the following implication holds:

$$f \in \mathrm{Ker}_h \mathcal{Q}_{\lambda_k} \quad \Rightarrow \quad \begin{cases} \partial_x f = X_k g \ \text{ with } \ g \in \mathcal{M}^s_{\lambda_{h-1,k-1}} \\ \partial_i f = 0,\ 1 \le i \le k \\ \langle u_i, \partial_j \rangle f = 0,\ 1 \le i < j \le k \end{cases} \tag{5.10}$$

In order to investigate the invertibility of this system, we proceed as follows. Consider the inhomogeneous systems of equations of the following type:

$$\begin{cases} \partial_0 f = g_0 \\ \partial_1 f = g_1 \\ \qquad \vdots \\ \partial_k f = g_k \end{cases}$$

with functions $f, g_0, \ldots, g_k : \mathbb{R}^m \to \mathbb{C}_m$. The study of such systems of equations for several Dirac operators is a complicated problem. In [24] these systems are successfully studied only in case of two and three Dirac operators. Therefore, put $k = 2$ and $(u_0, u_1, u_2) = (x, u, v)$. The compatibility conditions, i.e. conditions that have to be satisfied by g_0, g_1, g_2 in order to make the system solvable, are given by

$$\partial_i(\partial_j g_l + \partial_l g_j) = \{\partial_j, \partial_l\} g_i \tag{5.11}$$

with $0 \leq i, j, l \leq 2$. This is a restatement of the so-called radial algebra relations

$$[\partial_i, \{\partial_j, \partial_l\}] = 0, \quad 0 \leq i, j, l \leq 2.$$

In the case of three variables, the system (5.10) reduces to

$$f \in \mathrm{Ker}_h \mathcal{Q}_{1,1} \quad \Rightarrow \quad \begin{cases} \partial_x f = (v - u\langle v, \partial_u\rangle)\, g = \widetilde{v} g \\ \partial_u f = 0 \\ \partial_v f = 0 \\ \langle u, \partial_v\rangle f = 0 \end{cases} \tag{5.12}$$

with

$$g \in \mathcal{M}_{h-1,1} = \mathcal{S}_{h-1,1} \oplus \langle u, \partial_x\rangle \mathcal{M}_h.$$

If $g_0 = X_2 g$ and $g_1 = g_2 = 0$, the compatibility conditions (5.11) are given by

$$\begin{array}{ccc} \partial_u\partial_v\widetilde{v}g = 0 & \partial_x\partial_u\widetilde{v}g = 0 & \partial_x\partial_v\widetilde{v}g = 0 \\ \partial_v\partial_u\widetilde{v}g = 0 & \langle\partial_u, \partial_x\rangle\widetilde{v}g = 0 & \langle\partial_v, \partial_x\rangle\widetilde{v}g = 0. \end{array}$$

By means of

$$\begin{aligned} \{\partial_u, \widetilde{v}\} &= (m + 2\mathbb{E}_v - 4)\langle v, \partial_u\rangle - v\partial_u \\ \{\partial_v, \widetilde{v}\} &= -(m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v) + v\partial_v + u\partial_u - 2\langle v, \partial_u\rangle\langle u, \partial_v\rangle, \end{aligned}$$

it is easily verified that the compatibility conditions are satisfied:

$$\begin{aligned} \partial_u\partial_v\widetilde{v}g &= \partial_u\{\partial_v, \widetilde{v}\}g = 0 \\ \partial_v\partial_u\widetilde{v}g &= \partial_v\{\partial_u, \widetilde{v}\}g = 0 \\ \partial_x\partial_u\widetilde{v}g &= \partial_x\{\partial_u, \widetilde{v}\}g = 0 \\ [\langle\partial_u, \partial_x\rangle, \widetilde{v}]g &= \langle v, \partial_u\rangle\partial_x g = 0 \\ \partial_x\partial_v\widetilde{v}g &= \partial_x\{\partial_v, \widetilde{v}\}g = 0 \\ [\langle\partial_v, \partial_x\rangle, \widetilde{v}]g &= (\partial_x - u\langle\partial_u, \partial_x\rangle)g = 0. \end{aligned}$$

This means that there exists a solution f for the following system:

$$\begin{cases} \partial_x f = \widetilde{v} g \\ \partial_u f = 0 \\ \partial_v f = 0 \end{cases}$$

It follows from $\partial_u f = \partial_v f = 0$ that the polynomial f is $\mathcal{M}_{1,1}$-valued but not necessarily $\mathcal{S}_{1,1}$-valued. Since $\mathcal{M}_{1,1} = \mathcal{S}_{1,1} \oplus \langle v, \partial_u \rangle \mathcal{M}_1$, we use the projection operator

$$\Pi := \mathbf{1} - (\mathbb{E}_u - \mathbb{E}_v + 2)^{-1} \langle v, \partial_u \rangle \langle u, \partial_v \rangle : \mathcal{M}_{1,1} \to \mathcal{S}_{1,1}.$$

Referring once again to chapter 6 for a similar proof, the projection $\Pi(f)$ satisfies the extended system in (5.12):

$$\begin{cases} \partial_x \Pi(f) = \widetilde{v} g \\ \partial_u \Pi(f) = 0 \\ \partial_v \Pi(f) = 0 \\ \langle u, \partial_v \rangle \Pi(f) = 0 \end{cases}$$

Since $g \neq 0$, we have $\Pi(f) \neq 0$.

5.3.3 Decomposition of $\mathrm{Ker}_h \mathcal{Q}_{1,1}$

Summarising, two type A-solutions and two type B-solutions can be embedded in $\mathrm{Ker}_h \mathcal{Q}_{1,1}$ as follows:

$$\begin{aligned} \mathbf{1} : \mathcal{S}_{h,1,1} &\hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{1,1} \\ \langle v, \partial_x \rangle - \langle u, \partial_x \rangle \langle v, \partial_u \rangle : \mathcal{S}_{h+1,1} &\hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{1,1} \\ \varphi_1 : \mathcal{S}_{h-1,1} &\hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{1,1} \\ \varphi_2 : \mathcal{M}_h &\hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{1,1} \end{aligned}$$

An explicit expression for φ_1 and φ_2 is given in chapter 10. At this point, it is not obvious that there are no *more* irreducible summands in $\mathrm{Ker}_h \mathcal{Q}_{1,1}$, other than $\mathcal{S}_{h,1,1}$, $\big(\langle v, \partial_x \rangle - \langle u, \partial_x \rangle \langle v, \partial_u \rangle\big)\mathcal{S}_{h+1,1}$, $\varphi_1 \mathcal{S}_{h-1,1}$ and $\varphi_2 \mathcal{M}_h$. We prove this by verifying the dimensions, which we can prove in all generality.

Proposition 14. *For integers h, k with $h \geq k > 1$, one has*

$$\dim \mathrm{Ker}_h \mathcal{Q}_{\lambda_k} = \dim \mathcal{S}_{\lambda_{h,k}} \oplus \dim \mathcal{S}_{\lambda_{h+1,k-1}} \oplus \dim \mathcal{S}_{\lambda_{h-1,k-1}} \oplus \dim \mathcal{S}_{\lambda_{h,k-2}}.$$

Proof. Because the higher spin operators $\mathcal{Q}_{\lambda_k}$ are surjective, it follows that

$$\mathrm{Ker}_h \mathcal{Q}_{\lambda_k} \cong \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \bmod \mathcal{P}_{h-1}(\mathbb{R}^m, \mathcal{S}_{\lambda_k})$$

whence, using (2.44),

$$\begin{aligned}
\dim \mathrm{Ker}_h \mathcal{Q}_{\lambda_k} &= \dim \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) - \dim \mathcal{P}_{h-1}(\mathbb{R}^m, \mathcal{S}_{\lambda_k}) \\
&= \left[\begin{pmatrix} h+m-1 \\ h \end{pmatrix} - \begin{pmatrix} h+m-2 \\ h-1 \end{pmatrix} \right] \dim \mathcal{S}_{\lambda_k} \\
&= \begin{pmatrix} h+m-2 \\ h \end{pmatrix} \dim \mathcal{S}_{\lambda_k} \\
&= \dim \mathcal{P}_h(\mathbb{R}^{m-1}, \mathcal{S}_{\lambda_k}).
\end{aligned}$$

On the other hand, we can calculate $\dim \mathrm{Ker}_h \mathcal{Q}_{\lambda_k}$ as sum of the dimensions of its irreducible summands using (2.45):

$$\begin{aligned}
&\dim \mathcal{S}_{\lambda_{h,k}} \oplus \dim \mathcal{S}_{\lambda_{h+1,k-1}} \oplus \mathcal{S}_{\lambda_{h-1,k-1}} \oplus \dim \mathcal{S}_{\lambda_{h,k-2}} \\
=\ & 2^n \begin{pmatrix} h+2n-1 \\ h \end{pmatrix} \frac{(2n-2k+2)(2n+1)!}{k!(2n-k+2)!} \\
=\ & \dim \mathrm{Ker}_h \mathcal{Q}_{\lambda_k}.
\end{aligned}$$

□

Combining these results, we conclude

Proposition 15. *For all integers $h > 1$, one has*

$$\mathrm{Ker}_h \mathcal{Q}_{1,1} \cong \mathcal{S}_{h,1,1} \oplus \mathcal{S}_{h+1,1} \oplus \mathcal{S}_{h-1,1} \oplus \mathcal{M}_h.$$

It is convenient to write this in short as

$$\mathrm{Ker}_h \mathcal{Q}_{1,1} \cong \mathcal{M}^s_{h,1,1} \oplus \mathcal{M}_{h-1,1}. \tag{5.13}$$

In Figure 5.4 the decomposition into $\mathrm{Spin}(m)$-irreducibles of the space $\mathrm{Ker}_h \mathcal{Q}_{1,1}$ is visualised. The irreducible modules are denoted by their highest weight only.

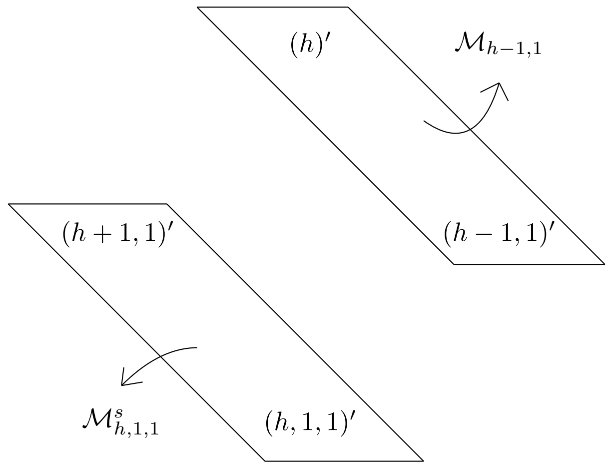


Figure 5.4: The decomposition of $\mathrm{Ker}_h\,\mathcal{Q}_{1,1}$.

Chapter 6

Construction of $\mathcal{Q}_{k,l}$

In this chapter, we give the explicit expression in Clifford analysis of the operator playing the fundamental role in this thesis: the unique (up to a multiplicative constant) elliptic conformally invariant first-order differential operator

$$\mathcal{Q}_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l});\ f(x;u,v) \mapsto \mathcal{Q}_{k,l} f(x;u,v).$$

It was mentioned before that the construction of the operator $\mathcal{Q}_{k,l}$ generalises the construction of the Rarita-Schwinger operator $\mathcal{R}_k$ in chapter 4. It is also similar to the construction of the operator $\mathcal{Q}_{\lambda_k}$ that acts on the space of spinor-valued differential forms in the previous chapter. Therefore we have postponed proofs of properties of these higher spin Dirac operators until this chapter. In the case of $\mathcal{Q}_{k,l}$ certain complications arise that were not yet visible in the cases of $\mathcal{R}_k$ and $\mathcal{Q}_{\lambda_k}$, which makes investigating the higher spin Dirac operator $\mathcal{Q}_{k,l}$ worth the effort.

In the first section, we prove the analogue of Proposition 12 in chapter 5, which can also be seen as the generalisation to two variables of the refinement of the monogenic Fischer decomposition of harmonical polynomials (3.6), given by $\mathcal{H}_k \otimes \mathbb{S} = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}$. As described in chapter 4, the latter decomposition gives rise to the Rarita-Schwinger operator $\mathcal{R}_k$, together with a so-called dual twistor operator. The three conformally invariant first-order differential operators arising from the decomposition of $\mathcal{H}_{k,l} \otimes \mathbb{S}$, among which the operator $\mathcal{Q}_{k,l}$ and two dual twistor operators, are defined in section 6.2. Finally, in the last section we give a proposition that will be essential in the construction of the space of homogeneous polynomial null solutions of $\mathcal{Q}_{k,l}$.

6.1 Decomposition of $\mathcal{H}_{k,l} \otimes \mathbb{S}$

Before proving the generalisation of the refined Fischer decomposition in one variable, recall the following notations. If V is a finite-dimensional representation (of a Lie group or Lie algebra) and $\mathbb{V}_\lambda$ an irreducible representation with highest weight λ, then the multiplicity of $\mathbb{V}_\lambda$ in V is denoted $n_\lambda(V)$ and the multiplicity of a weight μ in $\mathbb{V}_\lambda$ is denoted $m_\mu(\lambda)$.

Proposition 16. *For any pair of integers $k > l > 0$, one has*

$$(k,l) \otimes (0)' \;=\; (k,l)' \oplus (k,l-1)' \oplus (k-1,l)' \oplus (k-1,l-1)'. \tag{6.1}$$

If $k = l > 0$, one has

$$(k,k) \otimes (0)' \;=\; (k,k)' \oplus (k,k-1)' \oplus (k-1,k-1)'. \tag{6.2}$$

In case $l = 0$ this reduces to

$$(k) \otimes (0)' \;=\; (k)' \oplus (k-1)'. \tag{6.3}$$

Proof. We use Proposition 5 to prove (6.1). Recall that $\lambda = (k,l)$ is the highest weight for $\mathcal{H}_{k,l}$ and $\mu = (0)'$ is the highest weight for $\mathbb{S}$. Let ν be a dominant integral weight corresponding to one or more vector spaces in the decomposition $\mathcal{H}_{k,l} \otimes \mathbb{S}$, i.e. the multiplicity $n_\nu(\mathcal{H}_{k,l} \otimes \mathbb{S}) > 0$. Then, by Proposition 5, there is a weight s of $\mathbb{S}$ such that $\nu = \lambda + s$ and $n_\nu(\mathcal{H}_{k,l} \otimes \mathbb{S}) \leq m_s(\mathbb{S}) = 1$. This means that

$$n_\nu(\mathcal{H}_{k,l} \otimes \mathbb{S}) = 1.$$

All possible weights ν are given by

$$\nu = (k \pm \frac{1}{2}, l \pm \frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}).$$

Because ν is dominant integral, we only have to deal with the following cases: $\nu = (k,l)'$, $\nu = (k-1,l)'$, $\nu = (k,l-1)'$ and $\nu = (k-1,l-1)'$. The representations corresponding to these highest weights occur exactly once in $\mathcal{H}_{k,l} \otimes \mathbb{S}$. Note that the representation corresponding to $(k,l)'$ is the Cartan product of the tensor product $\mathcal{H}_{k,l} \otimes \mathbb{S}$ and always occurs with multiplicity 1. It is instructive to show this explicitly, e.g. for $\nu = (k,l-1)'$, using Klimyk's formula in Theorem 7:

$$\begin{aligned} n_\nu(\mathcal{H}_{k,l} \otimes \mathbb{S}) &= \sum_{w \in W} \operatorname{sgn}(w) m_{\nu+\delta-w(\lambda+\delta)}(\mathbb{S}) \\ &= m_{\nu+\delta-\mathbf{1}(\lambda+\delta)}(\mathbb{S}) = m_{(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})}(\mathbb{S}) = 1. \end{aligned}$$

Indeed, $w = \mathbf{1}$ is the only element of the Weyl group W leading to a non-trivial contribution. The other possibilities for ν are treated similarly. □

As stated in section 2.3.6, the spaces $\mathcal{H}_{k,l}$ and $\mathcal{S}_{k,l}$ are irreducible representations of the Spin group, with highest weights (k,l) and $(k,l)'$, respectively.

Corollary 2. *For any pair of integers $k > l > 0$, one has*

$$\mathcal{H}_{k,l} \otimes \mathbb{S} \;\cong\; \mathcal{S}_{k,l} \oplus \mathcal{S}_{k,l-1} \oplus \mathcal{S}_{k-1,l} \oplus \mathcal{S}_{k-1,l-1}.$$

If $k = l > 0$, one has

$$\mathcal{H}_{k,k} \otimes \mathbb{S} \;\cong\; \mathcal{S}_{k,k} \oplus \mathcal{S}_{k,k-1} \oplus \mathcal{S}_{k-1,k-1}.$$

6.1.1 Generalisation of the monogenic Fischer decomposition

Note that (6.3) is precisely the monogenic Fischer decomposition for spinor-valued harmonic polynomials:

$$\mathcal{H}_k \otimes \mathbb{S} = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}.$$

We will generalise this result to the case of spinor-valued simplicial harmonic polynomials $\mathcal{H}_{k,l} \otimes \mathbb{S}$ with $k > l > 0$. The special case $k = l$ is discussed at the end of this section. Proposition 16 states how this vector space decomposes into Spin(m)-irreducible summands, which implies the existence of certain maps ε that embed each of the spaces $\mathcal{S}_{k,l}$, $\mathcal{S}_{k-1,l}$, $\mathcal{S}_{k,l-1}$ and $\mathcal{S}_{k-1,l-1}$ (for appropriate k and l) into the space $\mathcal{H}_{k,l} \otimes \mathbb{S}$. To ensure that indeed $\varepsilon\mathcal{S}_{p,q} \hookrightarrow \mathcal{H}_{k,l} \otimes \mathbb{S}$, the conditions of Definition 1 in section 2.3.6 have to be satisfied.

Clearly, $\mathcal{S}_{k,l} \hookrightarrow \mathcal{H}_{k,l} \otimes \mathbb{S}$ is the trivial embedding. Also, it is easily verified that

$$u : \mathcal{S}_{k-1,l} \hookrightarrow \mathcal{H}_{k,l} \otimes \mathbb{S}.$$

In order to embed the space $\mathcal{S}_{k,l-1}$ in $\mathcal{H}_{k,l} \otimes \mathbb{S}$, it seems obvious to start from the basic invariant v, which is an embedding map of homogeneity degree $(0,1)$ in (u,v). However, this approach fails because $\langle u, \partial_v \rangle \big(v\mathcal{S}_{k,l-1}\big) = u\mathcal{S}_{k,l-1} \neq 0$. In order to obtain the correct embedding map, it suffices to project onto the kernel of the operator $\langle u, \partial_v \rangle$, which can be done by fixing c_1 in the following expression:

$$c_1 v - u\langle v, \partial_u \rangle : \mathcal{S}_{k,l-1} \hookrightarrow \mathcal{H}_{k,l} \otimes \mathbb{S}.$$

Indeed, since $\langle u, \partial_v\rangle^2 v\mathcal{S}_{k,l-1} = 0$, there are only two terms in the above embedding factor, and for $c_1 = k - l + 1$ every condition in Definition 1 is satisfied. Similarly, the last embedding map can be found as a suitable projection of a linear combination of uv and vu, and is given by

$$c_2\langle u, v\rangle - c_3 vu - |u|^2\langle v, \partial_u\rangle : \mathcal{S}_{k-1,l-1} \hookrightarrow \mathcal{H}_{k,l} \otimes \mathbb{S}$$

with $c_2 = m + 2k - 4$ and $c_3 = -(m + k + l - 4)$. Using Euler operators, this can be summarised as follows:

Theorem 21 (Simplicial monogenic Fischer decomposition of simplicial harmonics of two vector variables)**.** *For any pair of integers $k > l > 0$, one has*

$$\mathcal{H}_{k,l} \otimes \mathbb{S} \;=\; \mathcal{S}_{k,l} \;\oplus\; \widetilde{v}\mathcal{S}_{k,l-1} \;\oplus\; u\mathcal{S}_{k-1,l} \;\oplus\; \widetilde{\langle u, v\rangle}\mathcal{S}_{k-1,l-1}$$

where the embedding maps are defined as

$$\begin{aligned} \widetilde{v} \;&:=\; v(\mathbb{E}_u - \mathbb{E}_v) - u\langle v, \partial_u\rangle \\ \widetilde{\langle u, v\rangle} \;&:=\; \langle u, v\rangle(m + 2\mathbb{E}_u - 2) + vu(m + \mathbb{E}_u + \mathbb{E}_v - 2) - |u|^2\langle v, \partial_u\rangle. \end{aligned}$$

It can be verified that the second embedding factor can be written as

$$\widetilde{\langle u, v\rangle} = \frac{1}{2}\Big(\widetilde{v}u(m + 2\mathbb{E}_v - 2) - u\widetilde{v}(m + 2\mathbb{E}_u)\Big)(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1}.$$

Remark 21. *The symbol '$\sim$' in $\widetilde{v}$ and $\widetilde{\langle u, v\rangle}$ should not be confused with the anti-automorphism on $\mathbb{C}_m$ denoting the reversion.*

The summands in the decomposition in Theorem 21 are orthogonal with respect to the Fischer inner product on $\mathcal{P}(\mathbb{R}^{2m}, \mathbb{C}_m)$:

$$(f, g)_{(u,v)} \;=\; \big[f^\dagger(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})g\big]\Big|_{u=0,v=0} \tag{6.4}$$

where $f^\dagger(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the differential operator obtained by replacing $(u_1, \ldots, u_m)$, $(v_1, \ldots, v_m) \in \mathbb{R}^m$ in $f^\dagger(u, v)$ by $(\partial_{u_1}, \ldots, \partial_{u_n})$, $(\partial_{v_1}, \ldots, \partial_{v_n})$, respectively. The Fischer adjoints of the vector variables u and v with respect to this inner product are given by ∂_u and ∂_v, respectively. In the present case, f, g are polynomials in $\mathcal{H}_{k,l} \otimes \mathbb{S}$.

Remark 22. *Recall from (3.6) that the Howe-dual pair of the Fischer decomposition was given by* $(\mathrm{Pin}(m), \mathfrak{osp}(1|2))$*, and from Remark 5 that* $\mathfrak{osp}(1|4) \cong \mathrm{Alg}\{u, v, \partial_u, \partial_v\}$*. Explicit expressions and relations between the operators that generate the Lie superalgebra* $\mathfrak{osp}(1|4)$ *and its even part, i.e. the symplectic algebra* $\mathfrak{sp}(4)$*, can be found in [72]. It is not difficult to see that the Howe-dual pair of the Fischer decomposition in Theorem 21 equals* $(\mathrm{Pin}(m), \mathfrak{osp}(1|4))$*. See also [37].*

Remark 23. *In [72] the following monogenic Fischer decomposition in two vector variables is proved: if $m \geq 3$, then*

$$\mathcal{P}(\mathbb{R}^{2m}, V) \cong \bigoplus_{l=0}^{\infty} \mathcal{R}_l(u, v) \otimes \mathcal{M}(\mathbb{R}^{2m}, V)$$

with $V = \mathbb{C}_m$ or $V = \mathbb{S}$ and $\mathcal{R}_l(u, v)$ the subspace of polynomials in the radial algebra (which is isomorphic to $\mathrm{Alg}\{u, v\}$*) with total degree of homogeneity equal to l. The monogenic Fischer decomposition in two vector variables is much more complicated than the one-variable case, which can be illustrated with the following example: the counterpart of Δ_u in two variables is given by* $\mathrm{span}\{\Delta_u, \Delta_v, \langle \partial_u, \partial_v \rangle\}$*, and the counterpart of ∂_u in two variables equals* $\mathrm{span}\{\partial_u, \partial_v\}$*. However, we do not have that* $(\mathrm{span}\{\partial_u, \partial_v\})^2$ *equals the space* $\mathrm{span}\{\Delta_u, \Delta_v, \langle \partial_u, \partial_v \rangle\}$*, because $\partial_u \partial_v \notin$* $\mathrm{span}\{\Delta_u, \Delta_v, \langle \partial_u, \partial_v \rangle\}$*. We refer the interested reader to [72] for more information.*

6.1.2 Projection operators

An explicit expression for the projection operators on each of the summands in $\mathcal{H}_{k,l} \otimes \mathbb{S}$ can be obtained as follows. Suppose $\psi \in \mathcal{H}_{k,l} \otimes \mathbb{S}$. According to Theorem 21, there exist polynomials $\psi_{p,q} \in \mathcal{S}_{p,q}$ such that

$$\psi \;=\; \psi_{k,l} + \widetilde{v}\psi_{k,l-1} + u\psi_{k-1,l} + \widetilde{\langle u, v \rangle}\psi_{k-1,l-1}. \tag{6.5}$$

Using the following results:

$$\begin{aligned}
\{\partial_v, \widetilde{v}\} \;&=\; -\,(m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v) + v\partial_v + u\partial_u - 2\langle v, \partial_u \rangle \langle u, \partial_v \rangle \\
\big[\partial_v, \widetilde{\langle u, v \rangle}\big] \;&=\; -\,u(m + 2\mathbb{E}_v - 2)(m + \mathbb{E}_u + \mathbb{E}_v - 1) + vu\partial_v - |u|^2 \partial_u \\
[\partial_u, \widetilde{\langle u, v \rangle}] \;&=\; \big(v(m + 2\mathbb{E}_u - 2) - 2u\langle v, \partial_u \rangle\big)(m + \mathbb{E}_u + \mathbb{E}_v - 1) - uv\partial_u,
\end{aligned}$$

the action of ∂_v on (6.5) annihilates two summands and leads to

$$\begin{aligned}
\partial_v \psi \;=\; &-\,(m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v)\psi_{k,l-1} \\
&-\, u(m + 2\mathbb{E}_v - 2)(m + \mathbb{E}_u + \mathbb{E}_v - 1)\psi_{k-1,l-1}.
\end{aligned} \tag{6.6}$$

Acting again with ∂_u, we find

$$\begin{aligned}\psi_{k-1,l-1} &= (m+2\mathbb{E}_u)^{-1}(m+2\mathbb{E}_v-2)^{-1}(m+\mathbb{E}_u+\mathbb{E}_v-1)^{-1}\partial_u\partial_v\psi \\ &= \frac{\partial_u\partial_v\psi}{(m+2k-2)(m+2l-4)(m+k+l-3)}.\end{aligned}$$

This gives rise to a projection operator on the last summand

$$\pi_4 : \mathcal{H}_{k,l}\otimes\mathbb{S}\to\mathcal{S}_{k-1,l-1}$$

defined as

$$\pi_4(\psi) := (m+2\mathbb{E}_u)^{-1}(m+2\mathbb{E}_v-2)^{-1}(m+\mathbb{E}_u+\mathbb{E}_v-1)^{-1}\partial_u\partial_v\psi. \qquad (6.7)$$

Substituting the expression for $\psi_{k-1,l-1}$ in (6.6), we find

$$\begin{aligned}\psi_{k,l-1} &= -(m+2\mathbb{E}_v-2)^{-1}(\mathbb{E}_u-\mathbb{E}_v)^{-1}\left(\partial_v+(m+2\mathbb{E}_u-2)^{-1}u\partial_u\partial_v\right)\psi \\ &= -\frac{1}{(k-l+1)(m+2l-4)}\left(\mathbf{1}+\frac{u\,\partial_u}{m+2k-2}\right)\partial_v\psi\end{aligned}$$

which leads to a second projection operator

$$\pi_2 : \mathcal{H}_{k,l}\otimes\mathbb{S}\to\mathcal{S}_{k,l-1}$$

defined as

$$\pi_2(\psi) := -(m+2\mathbb{E}_v-2)^{-1}(\mathbb{E}_u-\mathbb{E}_v)^{-1}\left(\partial_v+(m+2\mathbb{E}_u-2)^{-1}u\partial_u\partial_v\right)\psi. \qquad (6.8)$$

Finally, using the previous results, the action of ∂_u on (6.5) leads to

$$\begin{aligned}\psi_{k-1,l} = -(m+2\mathbb{E}_u)^{-1}\Bigg[&(\mathbb{E}_u-\mathbb{E}_v+2)^{-1}\langle v,\partial_u\rangle\Big(\mathbf{1}+(m+2\mathbb{E}_u-2)^{-1}u\partial_u\Big)\partial_v \\ &-(m+2\mathbb{E}_u)^{-1}(m+2\mathbb{E}_v-4)^{-1}\Big(v(m+2\mathbb{E}_u-2)-2u\langle v,\partial_u\rangle\Big)\partial_u\partial_v+\partial_u\Bigg]\psi\end{aligned}$$

which can be written as

$$\begin{aligned}\psi_{k-1,l} = -\frac{1}{m+2k-2}\Bigg[&\left(\mathbf{1}+\left(\frac{k-l}{k-l+1}\right)\frac{v\,\partial_v}{m+2l-4}\right)\partial_u \\ &+\frac{1}{k-l+1}\left(\mathbf{1}+\frac{u\,\partial_u}{m+2l-4}\right)\langle v,\partial_u\rangle\partial_v\Bigg]\psi.\end{aligned}$$

This defines a third projection operator

$$\pi_3 : \mathcal{H}_{k,l} \otimes \mathbb{S} \to \mathcal{S}_{k-1,l}$$

as

$$\pi_3(\psi) := -(m+2\mathbb{E}_u)^{-1}\Bigg[(\mathbb{E}_u - \mathbb{E}_v + 2)^{-1}\langle v, \partial_u\rangle\Big(\mathbf{1} + (m+2\mathbb{E}_u-2)^{-1}u\partial_u\Big)\partial_v - (m+2\mathbb{E}_u)^{-1}(m+2\mathbb{E}_v-4)^{-1}\Big(v(m+2\mathbb{E}_u-2) - 2u\langle v,\partial_u\rangle\Big)\partial_u\partial_v + \partial_u\Bigg]\psi. \tag{6.9}$$

The projection operator on the first summand $\mathcal{S}_{k,l}$ in the decomposition of $\mathcal{H}_{k,l} \otimes \mathbb{S}$ is then given by

$$\pi_1 := \mathbf{1} - \widetilde{v}\pi_2 - u\pi_3 - \langle \widetilde{u,v}\rangle \pi_4. \tag{6.10}$$

A long calculation shows that this can be compactly written as

Lemma 19. *For $\psi \in \mathcal{H}_{k,l} \otimes \mathbb{S}$, one has*

$$\pi_1(\psi) = \left(1 + \frac{u\partial_u}{m+2k-2}\right)\left(1 + \frac{v\partial_v}{m+2l-4}\right)\left(1 + \frac{vu\partial_v\partial_u}{4(m+k+l-3)}\right)\psi. \tag{6.11}$$

Proof. By expanding the right-hand side of (6.11), one can verify that this equals (6.10). □

6.1.3 The case $k = l$

In the special case that $k = l > 0$, the summand $\mathcal{S}_{k-1,k}$ does not exist. This follows from the fact that $(k-1,k)'$ is not a dominant integral weight. Hence the following result, which can be proved analogously to the case $k > l$ of Theorem 21.

Theorem 22. *For all integers $k > 0$, one has*

$$\mathcal{H}_{k,k} \otimes \mathbb{S} = \mathcal{S}_{k,k} \oplus \widetilde{v}\mathcal{S}_{k,k-1} \oplus \langle \widetilde{u,v}\rangle\, \mathcal{S}_{k-1,k-1}$$

with the embedding maps defined as

$$\begin{aligned} \widetilde{v} &:= v - u\langle v, \partial_u\rangle \\ \langle \widetilde{u,v}\rangle &:= \langle v, u\rangle(m+2\mathbb{E}_u-2) + vu(m+2\mathbb{E}_u-2) - |u|^2\langle v,\partial_u\rangle. \end{aligned}$$

Note that the embedding factors are the same as in Theorem 21. Furthermore, the projection operators (6.7) and (6.8) remain the same. Because (6.9) does not exist, the projection operator on the first summand reduces to

$$\pi_1 := \mathbf{1} - \widetilde{v}\pi_2 - \widetilde{\langle u, v\rangle}\pi_4.$$

Remark 24. *In what follows, we will not make a difference between $k > l$ and $k = l$. It suffices to remember that, in the latter case, the summand corresponding to the highest weight $(k-1, k)'$ does not exist, and neither does the projection operator π_3 on this summand.*

6.2 Construction of the operator $\mathcal{Q}_{k,l}$

We now use Theorem 21 (or Theorem 22 in case $k = l$) to construct the higher spin Dirac operator $\mathcal{Q}_{k,l}$. Using the method of the twisted Dirac operator from section 3.3.3 or section 5.2, acting with ∂_x on $\mathcal{H}_{k,l} \otimes \mathbb{S}$-valued functions gives rise to a collection of invariant operators defined in Figure 6.1.

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S}) \arrow[r, bend left, "\partial_x"] \arrow[d, equal] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S}) \arrow[d, equal] \\
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \arrow[r, "\mathcal{Q}_{k,l}"] \arrow[ddr, "\widetilde{v}\mathcal{T}^{k,l}_{k,l-1}"] \arrow[ddddr, "u\mathcal{T}^{k,l}_{k-1,l}"'] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \\
\oplus & \oplus \\
\mathcal{C}^\infty(\mathbb{R}^m, \widetilde{v}\mathcal{S}_{k,l-1}) & \mathcal{C}^\infty(\mathbb{R}^m, \widetilde{v}\mathcal{S}_{k,l-1}) \\
\oplus & \oplus \\
\mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{S}_{k-1,l}) & \mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{S}_{k-1,l}) \\
\oplus & \oplus \\
\mathcal{C}^\infty(\mathbb{R}^m, \widetilde{\langle u, v\rangle}\mathcal{S}_{k-1,l-1}) & \mathcal{C}^\infty(\mathbb{R}^m, \widetilde{\langle u, v\rangle}\mathcal{S}_{k-1,l-1})
\end{tikzcd}$$

Figure 6.1: Introducing: operators $\mathcal{Q}_{k,l}$, $\mathcal{T}^{k,l}_{k,l-1}$ and $\mathcal{T}^{k,l}_{k-1,l}$.

There does not exist a conformally invariant first-order differential operator acting between $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$ and $\mathcal{C}^\infty(\mathbb{R}^m, \widetilde{\langle u, v\rangle}\mathcal{S}_{k-1,l-1})$. Again, this follows from Theorem 16 (see [38]), which states that there is no summand with highest weight $(k-1, l-1)'$ in the decomposition of the tensor product $\mathbb{R}^m \otimes \mathcal{S}_{k,l}$ into $\mathrm{Spin}(m)$-irreducibles.

The next lemma shows that this can also be verified through direct calculations in Clifford analysis.

Lemma 20. *If $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$, then $\partial_u \partial_v \partial_x f = 0$.*

Proof. The definition of the Euclidean inner product leads to

$$\partial_u \partial_v \partial_x f = -2\partial_u \langle \partial_v, \partial_x \rangle f - \partial_u \partial_x \partial_v f = 0,$$

because we have $\partial_u f = \partial_v f = 0$. □

Hence, it follows from (6.7) that

$$\pi_4(\partial_x f) = 0 \tag{6.12}$$

for every $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$. Furthermore,

$$\pi_2(\partial_x f) = -\frac{1}{(k-l+1)(m+2l-4)} \partial_v \partial_x f \tag{6.13}$$

$$\pi_3(\partial_x f) = -\frac{1}{(k-l+1)(m+2k-2)} \big((k-l+1)\partial_u + \langle v, \partial_u \rangle \partial_v\big)\partial_x f. \tag{6.14}$$

We introduce the short notation

$$\widetilde{\langle \partial_u, \partial_x \rangle} := \langle \partial_u, \partial_x \rangle (\mathbb{E}_u - \mathbb{E}_v + 1) + \langle v, \partial_u \rangle \langle \partial_v, \partial_x \rangle. \tag{6.15}$$

An explicit expression for the operators $\mathcal{Q}_{k,l}$, $\mathcal{T}^{k,l}_{k,l-1}$ and $\mathcal{T}^{k,l}_{k-1,l}$ in Figure 6.1 is then obtained using results of the previous section.

Definition 7. *For all integers $k \geq l \geq 0$ with $k > 0$, there are unique (up to a multiplicative constant) conformally invariant first-order differential operators $\mathcal{Q}_{k,l}$ defined by*

$$\mathcal{Q}_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) : f \mapsto \pi_1(\partial_x f)$$

and explicitly given by

$$\mathcal{Q}_{k,l} f = \left(\partial_x - \frac{2u\widetilde{\langle \partial_u, \partial_x \rangle}}{(k-l+1)(m+2k-2)} - \frac{2\widetilde{v}\langle \partial_v, \partial_x \rangle}{(k-l+1)(m+2l-4)} \right) f. \tag{6.16}$$

The ellipticity of this operator follows e.g. from [15], and the Spin(m)-invariance (recall that $\mathcal{Q}_{k,l}$ is even conformally invariant) can be expressed through the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \arrow[r, "\mathcal{Q}_{k,l}"] \arrow[d, "L_Q(s)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \arrow[d, "L_Q(s)"] \\
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \arrow[r, "\mathcal{Q}_{k,l}"'] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})
\end{tikzcd}$$

where

$$L_Q(s)\big(f(x;u,v)\big) := (H \otimes L)(s)\big(f(x;u,v)\big) = sf(\bar{s}xs; \bar{s}us, \bar{s}vs)$$

represents the simultaneous action of H on $x \in \mathbb{R}^m$ together with the L-action on the values $\mathcal{S}_{k,l}$.

Remark 25. *The expression for $\mathcal{Q}_{k,l}$ can be written as*

$$\begin{aligned}
\mathcal{Q}_{k,l}f &= \left(\mathbf{1} + \frac{u\partial_u}{m+2k-2} + \frac{v\partial_v}{m+2l-4} - 2\frac{u\langle v, \partial_u\rangle\partial_v}{(m+2k-2)(m+2l-4)}\right)\partial_x f. \\
&= \left(1 + \frac{u\partial_u}{m+2k-2}\right)\left(1 + \frac{v\partial_v}{m+2l-4}\right)\partial_x f,
\end{aligned}$$

using (6.11) and Lemma 20. In case $k = l > 0$, we have

$$\mathcal{Q}_{k,k}f = \left(\partial_x - \frac{2\widetilde{v}\langle\partial_v, \partial_x\rangle}{m+2k-4}\right)f = \left(\mathbf{1} + \frac{(v - u\langle v, \partial_u\rangle)\partial_v}{m+2k-4}\right)\partial_x f. \tag{6.17}$$

Finally, we obtain the Rarita-Schwinger operators $\mathcal{R}_k$ if $l = 0$:

$$\mathcal{Q}_{k,0} = \mathcal{R}_k = \left(\mathbf{1} + \frac{u\partial_u}{m+2k-2}\right)\partial_x.$$

Similar calculations lead to the so-called dual twistor operators, which are visualised as the diagonal arrows in Figure 6.1. We adopt the convention that each operator of twistor-type is denoted by means of the letter $\mathcal{T}$, together with upper and lower indices. The upper (resp. lower) indices denote the highest weight of the source (resp. target) space.

Definition 8. *For all integers $k \geq l > 0$, the dual twistor operators $\mathcal{T}^{k,l}_{k,l-1}$ are defined as the unique (up to a multiplicative constant) conformally invariant first-order differential operators*

$$\mathcal{T}^{k,l}_{k,l-1} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l-1}) : f \mapsto \pi_2(\partial_x f)$$

and explicitly given by

$$\mathcal{T}^{k,l}_{k,l-1} f = \frac{2}{(k-l+1)(m+2l-4)} \langle \partial_v, \partial_x \rangle f.$$

Definition 9. *For all integers $k > l \geq 0$, the dual twistor operators $\mathcal{T}^{k,l}_{k-1,l}$ are defined as the unique (up to a multiplicative constant) conformally invariant first-order differential operators*

$$\mathcal{T}^{k,l}_{k-1,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k-1,l}) : f \mapsto \pi_3(\partial_x f)$$

and explicitly given by

$$\mathcal{T}^{k,l}_{k-1,l} f = \frac{2}{(k-l+1)(m+2k-2)} \widetilde{\langle \partial_u, \partial_x \rangle} f.$$

In case $k = l > 0$, these operators do not exist.

These operators are called *dual* twistor operators, because there also exist twistor operators $\mathcal{T}^{k-1,l}_{k,l}$ and $\mathcal{T}^{k,l-1}_{k,l}$ acting in the opposite direction, defined as

$$\begin{aligned}
\mathcal{T}^{k-1,l}_{k,l} &: \mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{S}_{k-1,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) : uf \mapsto \pi_1(\partial_x u f) \\
\mathcal{T}^{k,l-1}_{k,l} &: \mathcal{C}^\infty(\mathbb{R}^m, \widetilde{v}\mathcal{S}_{k,l-1}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) : \widetilde{v} f \mapsto \pi_1(\partial_x \widetilde{v} f),
\end{aligned}$$

respectively.

Remark 26. *In what follows, we work with the dual twistor operators $\langle \partial_v, \partial_x \rangle$ and $\widetilde{\langle \partial_u, \partial_x \rangle}$, which equal, up to a multiplicative constant, the operators $\mathcal{T}^{k,l}_{k,l-1}$ and $\mathcal{T}^{k,l}_{k-1,l}$ defined above:*

$$\begin{aligned}
\langle \partial_v, \partial_x \rangle &= \frac{1}{2}(\mathbb{E}_u - \mathbb{E}_v)(m + 2\mathbb{E}_v - 2)\mathcal{T}^{k,l}_{k,l-1} \\
\widetilde{\langle \partial_u, \partial_x \rangle} &= \frac{1}{2}(\mathbb{E}_u - \mathbb{E}_v + 2)(m + 2\mathbb{E}_u)\mathcal{T}^{k,l}_{k-1,l}.
\end{aligned}$$

6.3 Useful properties

The following results can be proved by direct calculations.

Lemma 21. *One has*

$$[\widetilde{\langle \partial_u, \partial_x \rangle}, \langle \partial_v, \partial_x \rangle] = 0.$$

Proof. A simple calculation leads to the desired result:

$$\begin{aligned}
&[\langle \partial_u, \partial_x \rangle (\mathbb{E}_u - \mathbb{E}_v + 1), \langle \partial_v, \partial_x \rangle] + [\langle v, \partial_u \rangle \langle \partial_v, \partial_x \rangle, \langle \partial_v, \partial_x \rangle] \\
&= \langle \partial_u, \partial_x \rangle \langle \partial_v, \partial_x \rangle - \langle \partial_v, \partial_x \rangle \langle \partial_u, \partial_x \rangle = 0.
\end{aligned}$$

□

Proposition 17. *For any couple of integers $k > l > 0$, and for any function $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$, one has:*

$$\begin{aligned}
\pi_2\big(\partial_x u \widetilde{\langle \partial_u, \partial_x \rangle} f\big) &= 0 \\
\pi_3\big(\partial_x \widetilde{v} \langle \partial_v, \partial_x \rangle f\big) &= 0.
\end{aligned}$$

Proof. We prove the first statement. Up to a multiplicative constant, we have

$$\begin{aligned}
&\pi_2(\partial_x u \widetilde{\langle \partial_u, \partial_x \rangle} f) \\
&= \big(1 + u\partial_u (m + 2\mathbb{E}_u - 2)^{-1}\big) \partial_v \partial_x u \widetilde{\langle \partial_u, \partial_x \rangle} f \\
&= -2\big(1 + u\partial_u (m + 2\mathbb{E}_u - 2)^{-1}\big) \langle \partial_v, \partial_x \rangle u \widetilde{\langle \partial_u, \partial_x \rangle} f \\
&= -2\big(1 - (m + 2\mathbb{E}_u - 2)^{-1} (m + 2\mathbb{E}_u - 2)\big) \langle \partial_v, \partial_x \rangle u \widetilde{\langle \partial_u, \partial_x \rangle} f \ = \ 0.
\end{aligned}$$

The proof of the second statement is similar. □

The above proposition implies that there are no non-trivial first-order differential operators acting between $\mathcal{C}^\infty(\mathbb{R}^m, u\mathcal{S}_{k-1,l})$ and $\mathcal{C}^\infty(\mathbb{R}^m, \widetilde{v}\mathcal{S}_{k,l-1})$ (and vice versa); these non-existent operators have been visualised by dotted lines in Figure 6.2, where the double action of the Dirac operator on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$ is considered.

To end this chapter, we prove a very important result, which states that the dual twistor operators map null solutions of the operator $\mathcal{Q}_{k,l}$ to null solutions of related operators $\mathcal{Q}_{k,l-1}$ and $\mathcal{Q}_{k-1,l}$, respectively.

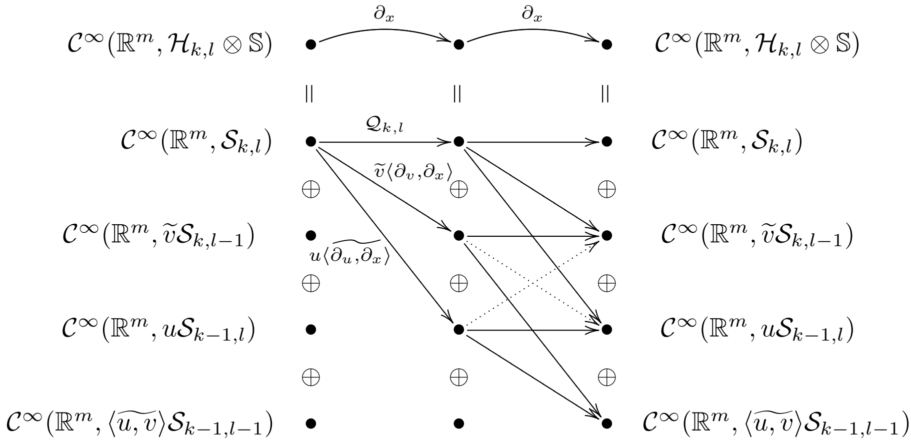


Figure 6.2: Action of ∂_x^2 on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$.

Proposition 18. *Let* $f \in \mathrm{Ker}_h \mathcal{Q}_{k,l}$.

(i) If $k \geq l > 0$, *then* $\langle\partial_v, \partial_x\rangle f \in \mathrm{Ker}_{h-1} \mathcal{Q}_{k,l-1}$.

(ii) If $k > l \geq 0$, *then* $\langle\widetilde{\partial_u, \partial_x}\rangle f \in \mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l}$.

Proof. Again, a straightforward calculation leads to the desired result. Let $c_1 = m + 2k - 2$ and $c_2 = m + 2l - 4$. For every $f \in \mathrm{Ker}_h \mathcal{Q}_{k,l}$, we have

$$\begin{aligned}
\langle\partial_v, \partial_x\rangle \mathcal{Q}_{k,l} f = 0 &\Leftrightarrow \langle\partial_v, \partial_x\rangle\big(c_1 c_2 + c_2 u\partial_u + c_1 v\partial_v - 2u\langle v, \partial_u\rangle\partial_v\big)\partial_x f = 0 \\
&\Leftrightarrow c_1(c_2 - 2)\partial_x\langle\partial_v, \partial_x\rangle f + (c_2 - 2)u\partial_u\partial_x\langle\partial_v, \partial_x\rangle f \\
&\qquad + c_1 v\partial_v\partial_x\langle\partial_v, \partial_x\rangle f - 2u\langle v, \partial_u\rangle\partial_v\partial_x\langle\partial_v, \partial_x\rangle f = 0 \\
&\Leftrightarrow \mathcal{Q}_{k,l-1}\langle\partial_v, \partial_x\rangle f = 0.
\end{aligned}$$

This may also be proved by considering the double action of the Dirac operator. Because ∂_x^2 is scalar, the following implication obviously holds:

$$f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \;\Rightarrow\; \partial_x^2 f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}).$$

Therefore, the projection on each of the other summands in the decomposition of $\mathcal{H}_{k,l} \otimes \mathbb{S}$ is zero. In particular, we have $\pi_2(\partial_x^2 f) = 0$, which, in combination with

Proposition 17 leads to the following identity (up to a suitable normalisation):

$$\widetilde{v}\langle\partial_v,\partial_x\rangle\mathcal{Q}_{k,l}+\widetilde{v}\mathcal{Q}_{k,l-1}\langle\partial_v,\partial_x\rangle=0. \tag{6.18}$$

This can be visualised by the parallelogram formed by the double arrows in Figure 6.3.

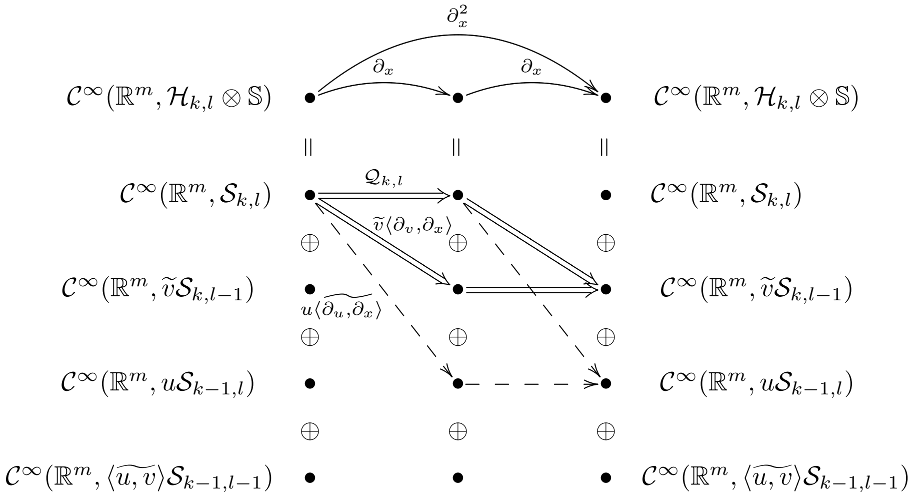


Figure 6.3: Parallelograms.

For $f\in\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$, the identity (6.18) reduces to $\widetilde{v}\mathcal{Q}_{k,l-1}\langle\partial_v,\partial_x\rangle f=0$, proving the first statement.

The calculations for proving the second statement are somewhat more technical and involved. On the one hand,

$$\begin{aligned}\mathcal{Q}_{k,l}f=0\ \Rightarrow\ &\langle\partial_u,\partial_x\rangle\mathcal{Q}_{k,l}f=0\\ \Leftrightarrow\ &(c_1-2)c_2\partial_x\langle\partial_u,\partial_x\rangle f+c_1v\partial_v\partial_x\langle\partial_u,\partial_x\rangle f+c_2u\partial_u\partial_x\langle\partial_u,\partial_x\rangle f\\ &-2u\langle v,\partial_u\rangle\partial_v\partial_x\langle\partial_u,\partial_x\rangle f+4\partial_x\langle v,\partial_u\rangle\langle\partial_v,\partial_x\rangle f=0. \end{aligned}\tag{6.19}$$

On the other hand, we have

$$\begin{aligned}\mathcal{Q}_{k,l}f=0\ \Rightarrow\ &\langle v,\partial_u\rangle\langle\partial_v,\partial_x\rangle\mathcal{Q}_{k,l}f=0\\ \Leftrightarrow\ &c_1(c_2-2)\partial_x\langle v,\partial_u\rangle\langle\partial_v,\partial_x\rangle f+(c_2-c_1)v\partial_u\partial_x\langle\partial_v,\partial_x\rangle f\\ &+c_2u\partial_u\partial_x\langle v,\partial_u\rangle\langle\partial_v,\partial_x\rangle f+(c_1-2)v\partial_v\partial_x\langle v,\partial_u\rangle\langle\partial_v,\partial_x\rangle f\\ &-c_1v\partial_u\partial_x\langle\partial_v,\partial_x\rangle f-2u\langle v,\partial_u\rangle\partial_v\partial_x\langle v,\partial_u\rangle\langle\partial_v,\partial_x\rangle f=0. \end{aligned}\tag{6.20}$$

Putting (6.19) and (6.20) together, we find

$$\begin{aligned}
&(c_1 - c_2)\langle \partial_u, \partial_x\rangle \mathcal{Q}_{k,l} f + 2\langle v, \partial_u\rangle\langle \partial_v, \partial_x\rangle \mathcal{Q}_{k,l} f = 0\\
&\qquad \Rightarrow \quad \mathcal{Q}_{k-1,l}\Big((k-l+1)\langle \partial_u, \partial_x\rangle + \langle v, \partial_u\rangle\langle \partial_v, \partial_x\rangle \Big) f = 0,
\end{aligned}$$

which leads to the desired statement. Invoking once more Proposition 17, this can also be proved by considering the parallelogram formed by the dashed lines in Figure 6.3; this leads to the identity (up to a suitable normalisation):

$$u\widetilde{\langle \partial_u, \partial_x\rangle}\mathcal{Q}_{k,l} + u\mathcal{Q}_{k-1,l}\widetilde{\langle \partial_u, \partial_x\rangle} = 0.$$

If $f \in \mathrm{Ker}_h \mathcal{Q}_{k,l}$, we find $u\mathcal{Q}_{k-1,l}\widetilde{\langle \partial_u, \partial_x\rangle} f = 0$, which concludes the proof. $\square$

Chapter 7

Results in Clifford Analysis

Now that we have constructed the operator $\mathcal{Q}_{k,l}$, which is the unique (up to a multiplicative constant) elliptic conformally invariant first-order differential operator acting as

$$\begin{aligned} \mathcal{Q}_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) &\rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \\ f(x;u,v) &\mapsto \mathcal{Q}_{k,l} f(x;u,v), \end{aligned}$$

we are ready to study some results of this higher spin Dirac operator. In section 7.1 we prove the generalisation of the (classical) Cauchy-Kowalewskaia extension (CK-extension) that was explained in chapter 3. By means of the conformal invariance of $\mathcal{Q}_{k,l}$, which is recalled in section 7.2, we construct in section 7.3 the fundamental solution corresponding to this higher spin Dirac operator. We end this chapter by proving the basic integral formulae for $\mathcal{Q}_{k,l}$: the theorems of Stokes and Cauchy-Pompeiu and the integral formula of Cauchy.

7.1 A generalised CK-extension

In section 3.2.4, the dimension of $\mathcal{M}_k = \mathrm{Ker}_k \partial_x$ was calculated by means of the (classical) CK-extension with respect to the operator ∂_x. In order to find the dimension of the vector space $\mathrm{Ker}_h \mathcal{Q}_{k,l}$, we generalise the CK-extension principle. The problem is formulated as follows: "Given a real-analytic function f^* in $\Omega^* \subset \mathbb{R}^{m-1}$, does there exist a function f in the kernel space of $\mathcal{Q}_{k,l}$ in an open neighbourhood Ω of Ω^* in $\mathbb{R}^m$ such that $f|_{\Omega^*} = f^*$?"

We begin with a lemma.

Lemma 22. *Acting on $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$, one has*

$$\pi_1 e_m f = 0 \;\Rightarrow\; f = 0.$$

This means that $\pi_1 e_m$ is invertible on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$.

Proof. If $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$, then $\pi_1 f = f$ and $e_m f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S})$. Hence,

$$\begin{aligned} \pi_1 e_m f = 0 &\Leftrightarrow [\pi_1, e_m] f + e_m \pi_1 f = 0 \\ &\Leftrightarrow [\pi_1, e_m] f = -e_m f \end{aligned}$$

which is a contradiction if $f \neq 0$. $\square$

Once again, we consider $\mathbb{R}^{m-1}$ as the hyperplane $x_m = 0$ in $\mathbb{R}^m$. This induces the following splitting of $\mathbb{R}^m$:

$$x = x^* + e_m x_m \in \mathbb{R}^m = \mathbb{R}^{m-1} \oplus \mathbb{R}$$

and the Dirac operator in $\mathbb{R}^{m-1}$ is given by $\partial_{x^*} = \sum_{i=1}^{m-1} e_i \partial_{x_i}$. As before, Ω denotes an open connected and x_m-normal neighbourhood of Ω^*. Define $\mathcal{Q}^*_{k,l} := \pi_1(\partial_{x^*})$ on $\mathbb{R}^{m-1}$. Then,

$$f \in \mathrm{Ker}_h \mathcal{Q}_{k,l} \;\Leftrightarrow\; \partial_{x_m} f = -\big(\pi_1(e_m)\big)^{-1} \mathcal{Q}^*_{k,l} f.$$

The function f has to satisfy the conditions $\partial_x f(x; u, v) = 0$ in Ω and

$$f(x^*, x_m; u, v)|_{x_m = 0} = f^*(x^*; u, v).$$

The unique solution of the first-order differential equation above is given by

$$f(x; u, v) = \exp\bigg(-x_m \big(\pi_1(e_m)\big)^{-1} \mathcal{Q}^*_{k,l}\bigg) f^*(x^*; u, v)$$

and is referred to as the *generalised CK-extension* of f^*. This CK-extension defines an isomorphism between the space of h-homogeneous polynomials on $\mathbb{R}^{m-1}$ taking values in $\mathcal{S}_{k,l}$, and the space of h-homogeneous polynomial null solutions of the operator $\mathcal{Q}_{k,l}$ on $\mathbb{R}^m$. In particular,

$$\dim \mathrm{Ker}_h \mathcal{Q}_{k,l} = \dim \mathcal{P}_h(\mathbb{R}^{m-1}, \mathcal{S}_{k,l}) = \dim \mathcal{P}_h(\mathbb{R}^{m-1}) \dim \mathcal{S}_{k,l}. \tag{7.1}$$

As the operator $\mathcal{Q}_{k,l}$ is surjective (see section 3.3.2), we have that

$$\mathrm{Ker}_h\, \mathcal{Q}_{k,l} \cong \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_{k,l}) \bmod \mathcal{P}_{h-1}(\mathbb{R}^m, \mathcal{S}_{k,l}) \tag{7.2}$$

whence

$$\begin{aligned}
\dim \mathrm{Ker}_h\, \mathcal{Q}_{k,l} &= \dim \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_{k,l}) - \dim \mathcal{P}_{h-1}(\mathbb{R}^m, \mathcal{S}_{k,l}) \\
&= \left[\begin{pmatrix} h+m-1 \\ h \end{pmatrix} - \begin{pmatrix} h+m-2 \\ h-1 \end{pmatrix} \right] \dim \mathcal{S}_{k,l} \\
&= \dim \mathcal{P}_h(\mathbb{R}^{m-1}) \dim \mathcal{S}_{k,l}.
\end{aligned}$$

This leads to the same result as in (7.1).

7.2 Conformal invariance

It has been mentioned many times before that the higher spin Dirac operator $\mathcal{Q}_{k,l}$ is conformally invariant (see e.g. [38, 65] and section 3.3.1). In order to construct the fundamental solution of the operator $\mathcal{Q}_{k,l}$ in the next section, we use a general theorem about conformally invariant first-order differential operators (Theorem 18) adapted to our case of operators acting on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$.

Let $\Omega \subset \mathbb{R}^m$ and consider the representation L of $\mathrm{Pin}(m)$ acting on the values $\mathcal{S}_{k,l}$. The action of the conformal group on a function $f \in \mathcal{C}^\infty(\Omega, \mathcal{S}_{k,l})$ is given by

$$\begin{aligned}
&(g \cdot f)(x) \\
&= |cx+d|^{-m+1} L\left(\frac{(cx+d)^\sim}{|cx+d|}\right) f((ax+b)(cx+d)^{-1}; u, v) \qquad (7.3)\\
&= \frac{(cx+d)^\sim}{|cx+d|^m} f((ax+b)(cx+d)^{-1}; \frac{(cx+d)u(cx+d)^\sim}{|cx+d|^2}, \frac{(cx+d)v(cx+d)^\sim}{|cx+d|^2})
\end{aligned}$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(m)$. Then also $g \cdot f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$. Due to the conformal invariance of $\mathcal{Q}_{k,l}$, Theorem 18 translates to

Theorem 23. *If $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$ is a null solution of $\mathcal{Q}_{k,l}$, so is the transformed function $g \cdot f$, with g an element of the conformal group.*

7.3 Fundamental solution of $\mathcal{Q}_{k,l}$

The fundamental solution of the Rarita-Schwinger operators was studied in [18] and, more recently, in [70].

Recall from section 3.2.5 that the Cauchy kernel $E(x)$ is defined as

$$E(x) = -\frac{1}{A_m}\frac{x}{|x|^m}. \tag{7.4}$$

As it satisfies, in distributional sense,

$$\partial_x E(x) = \delta(x),$$

the Cauchy kernel is the fundamental solution of the classical Dirac operator ∂_x. Inspired by Theorem 23, we will generalise this concept to the case of $\mathcal{Q}_{k,l}$. Here is an overview of the procedure.

Let $P_{k,l}$ be a polynomial in $\mathcal{S}_{k,l}$. In section 7.3.1 we prove that the polynomial $E_{k,l}(x; u, v)$, defined as

$$E_{k,l}(x; u, v) := C_{k,l}|x|^{-m+1}L(\frac{x}{|x|})P_{k,l}(u, v),$$

belongs to the kernel of $\mathcal{Q}_{k,l}$ and has a pointwise singularity in $x = 0$. As a next step, we determine the value of the constant $C_{k,l}$ in section 7.3.2. This goes by means of the theorem of distributions related to Riesz potentials in $\mathbb{R}^m$ ([45, 41]) and yields

$$C_{k,l} = -\frac{1}{A_m}\frac{(m + 2k - 2)(m + 2l - 4)}{(m - 2)(m - 4)}. \tag{7.5}$$

Note that in case $k = l = 0$, we have $\mathcal{Q}_{0,0} = \partial_x$ and $C_{0,0} = -A_m^{-1}$, which is in correspondence with (7.4). With this choice for the constant $C_{k,l}$, we prove that the polynomial $E_{k,l}(x; u, v)$ satisfies, in distributional sense,

$$\mathcal{Q}_{k,l}E_{k,l}(x; u, v) = \delta(x)P_{k,l}(u, v).$$

To make this relation independent of $P_{k,l} \in \mathcal{S}_{k,l}$, we introduce in section 7.3.3 the reproducing kernel on the space of simplicial monogenics.

7.3.1 The generalisation of the Cauchy kernel

Choosing $c = 1$ and $d = 0$ in (7.3), Theorem 23 implies the following result.

Lemma 23. *Let $C_{k,l}$ be a constant. For every $P \in \mathcal{S}_{k,l}$, the function*

$$E_{k,l}(x; u, v) \;=\; C_{k,l}|x|^{-m+1}L(\frac{x}{|x|})P(u, v) \tag{7.6}$$

is an element of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$. Furthermore, $E_{k,l}(x; u, v)$ belongs to the kernel of $\mathcal{Q}_{k,l}$ and has a singularity of degree $-m + 1$ in $x = 0$.

In order to find $C_{k,l}$, we proceed with explicit calculations.

Lemma 24. *If $P_{k,l}(u, v) \in \mathcal{S}_{k,l}$, the polynomial $xP_{k,l}(xux, xvx)$ is an element of $\mathcal{S}_{k,l}$ and the polynomial $P_{k,l}(xux, xvx)$ belongs to $\mathcal{H}_{k,l} \otimes \mathbb{S}$.*

Proof. Due to $\partial_u = x\partial_{xux}x$, we have

$$\partial_u xP_{k,l}(xux, xvx) = -x|x|^2\partial_{xux}P_{k,l}(xux, xvx) = 0,$$

because $\partial_u P_{k,l}(u, v) = 0$. In the same way, we have $\partial_v xP_{k,l}(xux, xvx) = 0$. Since $\langle xux, \partial_{xvx}\rangle = -|x|^4\langle u, \partial_v\rangle$, we also have $\langle u, \partial_v\rangle P_{k,l}(xux, xvx) = 0$. Thus, $xP_{k,l}(xux, xvx) \in \mathcal{S}_{k,l}$, whence it follows that $-|x|^2P_{k,l}(xux, xvx) \in x\mathcal{S}_{k,l} \subset \mathcal{H}_{k,l} \otimes \mathbb{S}$. This means that the polynomial $P_{k,l}(xux, xvx) \in \mathcal{H}_{k,l} \otimes \mathbb{S}$. □

Instead of taking P arbitrarily in $\mathcal{S}_{k,l}$, it is convenient to work with the following polynomial in $\mathcal{S}_{k,l}$ (see [73] for details):

$$P_{k,l}(u, v) := \langle u, \mathfrak{f}_1\rangle^{k-l}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^l I_{12}. \tag{7.7}$$

The vectors $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are Witt basis vectors in $\mathbb{C}^m$ as defined in (2.2), the idempotent $I_{12} = \mathfrak{f}_1\mathfrak{f}_1^\dagger\mathfrak{f}_2\mathfrak{f}_2^\dagger$ and

$$\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle := \det\begin{pmatrix} \langle u, \mathfrak{f}_1\rangle & \langle u, \mathfrak{f}_2\rangle \\ \langle v, \mathfrak{f}_1\rangle & \langle v, \mathfrak{f}_2\rangle \end{pmatrix}.$$

It is clear that $\langle u, \partial_v\rangle P_{k,l}(u, v) = 0$ and $\partial_u P_{k,l}(u, v) = \partial_v P_{k,l}(u, v) = 0$, since $\mathfrak{f}_1^2 = \mathfrak{f}_2^2 = 0$.

With the choice (7.7) for $P_{k,l}$, the right-hand side of (7.6) leads to

$$\begin{aligned} |x|^{-m+1}L(\frac{x}{|x|})P_{k,l}(u, v) \;&=\; x|x|^{-m-2(k+l)}P_{k,l}(xux, xvx) \\ &=\; x|x|^{-m-2(k+l)}\langle xux, \mathfrak{f}_1\rangle^{k-l}\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^l I_{12}. \end{aligned}$$

To show that this polynomial is a null solution of $\mathcal{Q}_{k,l} = \pi_1(\partial_x)$, we proceed in two steps: first, we calculate the action with ∂_x and afterwards we make the projection π_1. Using the identities

$$\begin{aligned}
\langle xux, \mathfrak{f}_1\rangle &= |x|^2\langle u, \mathfrak{f}_1\rangle - 2\langle u, x\rangle\langle x, \mathfrak{f}_1\rangle \\
\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle &= |x|^4\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle - 2|x|^2\langle x, u\rangle\langle x \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle \\
&\quad - 2|x|^2\langle x, v\rangle\langle u \wedge x, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle,
\end{aligned}$$

we find

$$\begin{aligned}
&\mathcal{Q}_{k,l}\left(|x|^{-m-2(k+l)} x P_{k,l}(xux, xvx)\right) \\
&= (m + 2(k+l))|x|^{-m-2(k+l)}\pi_1\big(P_{k,l}(xux, xvx)\big) \\
&\quad + |x|^{-m-2(k+l)}\pi_1\left(\partial_x\big(xP_{k,l}(xux, xvx)\big)\right). \qquad (7.8)
\end{aligned}$$

The calculation of the second term in the right-hand side requires once again the use of the identities above:

$$\begin{aligned}
&\partial_x\big(xP_{k,l}(xux, xvx)\big) \\
&= -(m + 2\mathbb{E}_x + x\partial_x)P_{k,l}(xux, xvx) \\
&= -(m + 4(k+l))P_{k,l}(xux, xvx) - x\partial_x P_{k,l}(xux, xvx) \qquad (7.9)
\end{aligned}$$

with

$$\begin{aligned}
x\partial_x P_{k,l}(xux, xvx) = \Big(&(k-l)\big[x\partial_x\langle xux, \mathfrak{f}_1\rangle\big]\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle \\
&+ l\big[x\partial_x\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle\big]\langle xux, \mathfrak{f}_1\rangle\Big)\langle xux, \mathfrak{f}_1\rangle^{k-l-1}\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l-1} I_{12}.
\end{aligned}$$

By means of

$$\begin{aligned}
\partial_x\langle xux, \mathfrak{f}_1\rangle I_{12} &= \big(2x\langle u, \mathfrak{f}_1\rangle - 2u\langle x, \mathfrak{f}_1\rangle\big)I_{12} \\
x\partial_x\langle xux, \mathfrak{f}_1\rangle I_{12} &= \big(-2\langle xux, \mathfrak{f}_1\rangle + 2ux\langle x, \mathfrak{f}_1\rangle\big)I_{12} \\
x\partial_x\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle I_{12} &= \Big(-4\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle + 2|x|^2 ux\langle x \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle \\
&\quad + 2|x|^2 vx\langle u \wedge x, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle\Big)I_{12},
\end{aligned}$$

the right-hand side of (7.9) can be rewritten as

$$
\begin{aligned}
&-(m+2(k+l))P_{k,l}(xux,xvx) - 2(k-l)u\Big(\langle x,\mathfrak{f}_1\rangle x P_{k-1,l}(xux,xvx)\Big)\\
&+\frac{2l}{k-l+1}\widetilde{v}\Big(\langle x,\mathfrak{f}_2\rangle x P_{k,l-1}(xux,xvx)\Big)\\
&-2l\langle x,\mathfrak{f}_1\rangle\big(vx\langle xux,\mathfrak{f}_2\rangle - ux\langle xvx,\mathfrak{f}_2\rangle\big)xP_{k-1,l-1}(xux,xvx).
\end{aligned}
$$

At this point we take the projection with π_1 into account. It follows from Proposition 21 in chapter 6 that $\pi_1(u\mathcal{S}_{k-1,l}) = 0$ and $\pi_1(\widetilde{v}\mathcal{S}_{k,l-1}) = 0$. Together with the relation

$$
\pi_1\Big((v\langle xux,\mathfrak{f}_2\rangle - u\langle xvx,\mathfrak{f}_2\rangle)\mathcal{S}_{k-1,l-1}\Big) = 0,
$$

which is not difficult to calculate, the result in (7.8) leads to

$$
\begin{aligned}
&\mathcal{Q}_{k,l}\left(|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx)\right)\\
&= -(m+2(k+l))|x|^{-m-2(k+l)}\pi_1\big(P_{k,l}(xux,xvx)\big)\\
&\quad + (m+2(k+l))|x|^{-m-2(k+l)}\pi_1\big(P_{k,l}(xux,xvx)\big) = 0.
\end{aligned}
$$

7.3.2 Calculating the constant $C_{k,l}$

Now consider, for $\alpha \in \mathbb{C}$, the function

$$
|x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx) \in \mathcal{C}^\infty(\mathbb{R}^m\setminus\{0\},\mathcal{S}_{k,l}),
$$

which, by the action of the operator $\mathcal{Q}_{k,l}$, is turned into

$$
\begin{aligned}
&\mathcal{Q}_{k,l}\left(|x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx)\right)\\
&= -(\alpha-2(k+l))|x|^{\alpha-2(k+l)}\pi_1\big(P_{k,l}(xux,xvx)\big)\\
&\quad - (m+2(k+l))|x|^{\alpha-2(k+l)}\pi_1\big(P_{k,l}(xux,xvx)\big)\\
&= -(\alpha+m)|x|^{\alpha-2(k+l)}\pi_1\big(P_{k,l}(xux,xvx)\big).
\end{aligned}
\tag{7.10}
$$

If we put $\alpha = -m$ then it is clear that $|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx)$ belongs to the kernel of $\mathcal{Q}_{k,l}$. Furthermore, it shows a pointwise singularity of homogeneity degree $-m+1$ at the origin $x=0$; in order to calculate the residue in $\alpha=-m$, we proceed as follows.

Riesz potentials

For $\Re(\alpha) > -m-1$, the function

$$x \mapsto |x|^{\alpha-2(k+l)} x P_{k,l}(xux, xvx)$$

belongs to $L_1^{loc}(\mathbb{R}^m, \mathcal{S}_{k,l})$ and so defines a distribution on the space $\mathcal{D}(\mathbb{R}^m, \mathcal{S}_{k,l})$ of test functions in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$ with compact support. It is considered now as the product of $|x|^{\alpha-2(k+l)}$ with the polynomial $xP_{k,l}(xux, xvx)$.

Let us recall some basic facts about distributions related to Riesz potentials in $\mathbb{R}^m$ (see [45, 41] and also [7, 8] for their generalisations in Clifford analysis). Consider for $\Re(\lambda) > -m$ the distribution $|x|^\lambda$ on $\mathbb{R}^m$ given by

$$\langle |x|^\lambda, \varphi \rangle = \int_{\mathbb{R}^m} |x|^\lambda \varphi(x) dx$$

with $\varphi \in \mathcal{D}(\mathbb{R}^m)$. The mapping $\lambda \to |x|^\lambda$ extends uniquely to a meromorphic mapping from $\mathbb{C}$ (i.e. holomorphic on $\mathbb{C}$ except on a set of isolated points) to the space of tempered distributions. The poles are located at $\{-m-2a, a \in \mathbb{N}\}$ and they are all simple. If $\lambda \neq -m-2a$, we have, in distributional sense, $\Delta_x |x|^\lambda = \lambda(\lambda+m-2)|x|^{\lambda-2}$. For $\Re(\lambda) > 0$, the Riesz potential I^λ is defined as

$$I^\lambda \varphi = \frac{\Gamma(\frac{m-\lambda}{2})}{\pi^{\frac{m}{2}} 2^\lambda \Gamma(\frac{\lambda}{2})} (\varphi * |x|^{-m+\lambda})$$

with $I^0\varphi = \varphi = \varphi * \delta$. It satisfies $\Delta_x I^\lambda = -I^{\lambda-2}$ in distributional sense. This means that $I^\lambda = -\Delta_x I^{\lambda+2}$; in this way one can extend I^λ to $\Re(\lambda) > -2$. In general, $I^\lambda = (-1)^b \Delta_x^b I^{\lambda+2b}$ ($b \in \mathbb{N}$). Note that the poles $\lambda = -2a$ of $\Gamma(\frac{\lambda}{2})$ cancel against the poles of $|x|^{-m+\lambda}$. Thus, $\lambda \mapsto I^\lambda$ can be analytically extended to a holomorphic function for $\lambda \neq m+2a$, which are precisely the poles of $\Gamma(\frac{m-\lambda}{2})$. In the points $\lambda = -2a$ of this extension, one obtains the distributions

$$I^{-2a} = (-1)^a \Delta_x^a \delta(x). \tag{7.11}$$

These results on Riesz potentials can be reformulated in terms of the distribution $|x|^{-m+\lambda}$. In this way the mapping $\lambda \mapsto |x|^{-m+\lambda}$ may be continued analytically to $\mathbb{C} \setminus \{-2a, a \in \mathbb{N}\}$. Its singularities are simple poles with corresponding residues given by

$$\begin{aligned}
\mathrm{Res}\left[|x|^{-m+\lambda}, \lambda=-2a\right] &= \mathrm{Res}\left[\frac{\pi^{\frac{m}{2}}2^{\lambda}\Gamma(\frac{\lambda}{2})}{\Gamma(\frac{m-\lambda}{2})}I^{\lambda}, \lambda=-2a\right] \\
&= \frac{\pi^{\frac{m}{2}}2^{-2a}}{\Gamma(\frac{m}{2}+a)}\mathrm{Res}\left[\Gamma(\frac{\lambda}{2}), \lambda=-2a\right]I^{-2a} \\
&= \frac{\pi^{\frac{m}{2}}2^{-2a+1}}{\Gamma(\frac{m}{2}+a)a!}\Delta_x^a\delta(x)
\end{aligned}$$

in view of (7.11) and

$$\begin{aligned}
\mathrm{Res}\left[\Gamma(\frac{\lambda}{2}), \lambda=-2a\right] &= \lim_{\lambda\to-2a}(\lambda+2a)\Gamma(\frac{\lambda}{2}) \\
&= 2\lim_{x\to-a}(a+x)\Gamma(x) \\
&= 2\,\mathrm{Res}\left[\Gamma(x), x=-a\right] = 2\frac{(-1)^a}{a!}.
\end{aligned}$$

In particular, the mapping $\alpha \mapsto |x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx)$ is holomorphic in $\mathbb{C}\setminus\{-m+2(k+l)-2a, a\in\mathbb{N}\}$. Because the polynomial $xP_{k,l}(xux,xvx)$ is homogeneous of degree $2(k+l)+1$ in x, the following poles are removable singularities:

$$-m, -m+2, \ldots, -m+2(k+l-2), -m+2(k+l-1).$$

This can be proved by calculating the residues. For example,

$$\begin{aligned}
&\mathrm{Res}\left[|x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx), \alpha=-m\right] \\
&= \lim_{\alpha\to-m}(\alpha+m)\,|x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx)
\end{aligned} \tag{7.12}$$

and we can write, by substituting $x=r\omega$ with $r=|x|$ and $\omega=\frac{x}{|x|}$,

$$|x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx) = r^{\alpha+1}\omega g(\omega;u,v)$$

where $g(\omega;u,v)$ is a polynomial, k-homogeneous in u and l-homogeneous in v and independent of α. As a consequence, the residue (7.12) equals zero. The same argument holds for the other residues.

We conclude that the mapping $\alpha \mapsto |x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx)$ is holomorphic in $\mathbb{C}\setminus\{-m-2a, a\in\mathbb{N}\}$.

The map ξ_x

It follows from the previous paragraph that the relation (7.10) remains valid, in distributional sense, for $\Re(\alpha) > -m-1$. Hence,

$$\begin{aligned}
&\mathcal{Q}_{k,l}\left(|x|^{-m-2(k+l)} x P_{k,l}(xux, xvx)\right) \\
&= -\lim_{\alpha\to -m}(\alpha+m)|x|^{\alpha-2(k+l)}\pi_1\big(P_{k,l}(xux,xvx)\big) \\
&= -\text{Res}\left[|x|^{\alpha-2(k+l)}, \alpha=-m\right]\pi_1\big(P_{k,l}(xux,xvx)\big) \\
&= -\text{Res}\left[|x|^{-m+\lambda}, \lambda=-2(k+l)\right]\pi_1\big(P_{k,l}(xux,xvx)\big) \\
&= -\frac{\pi^{\frac{m}{2}}2^{-2(k+l)+1}}{\Gamma(\frac{m}{2}+k+l)(k+l)!}\Delta_x^{k+l}\delta(x)\,\pi_1\big(P_{k,l}(xux,xvx)\big) \\
&= -\frac{\pi^{\frac{m}{2}}2^{-2(k+l)+1}}{\Gamma(\frac{m}{2}+k+l)(k+l)!}\Delta_x^{k+l}\pi_1\big(P_{k,l}(xux,xvx)\big)\delta(x) \qquad (7.13)
\end{aligned}$$

where the last step follows from elementary properties of distributions. Indeed, using the fact that $P_{k,l}(xux, xvx)$ is homogeneous of degree $2(k+l)$ in x, we have for all test functions φ

$$\begin{aligned}
\langle(\Delta_x^{k+l}\delta)(\pi_1 P_{k,l}),\varphi\rangle &= \langle\Delta_x^{k+l}\delta,(\pi_1 P_{k,l})\varphi\rangle \\
&= \langle\delta,\Delta_x^{k+l}(\pi_1 P_{k,l})\varphi+(\pi_1 P_{k,l})\Delta_x^{k+l}\varphi+\cdots\rangle \\
&= \langle\delta,\Delta_x^{k+l}(\pi_1 P_{k,l})\varphi\rangle \\
&= \langle\Delta_x^{k+l}(\pi_1 P_{k,l})\delta,\varphi\rangle,
\end{aligned}$$

because $\langle\delta,\varphi\rangle = \varphi(0)$ and $\Delta_x^{k+l}(\pi_1 P_{k,l})$ is the only term in the sum that does not depend on x.

Next, we calculate the expression $\Delta_x^{k+l}\pi_1\big(P_{k,l}(xux,xvx)\big)$ in (7.13), which equals $\pi_1\big(\Delta_x^{k+l}P_{k,l}(xux,xvx)\big)$. To this end, we introduce the following operator:

$$\begin{aligned}
\xi_x : \mathcal{S}_{k,l} &\rightarrow \mathcal{P}_{2(k+l)}\otimes\mathcal{H}_{k,l}\otimes\mathbb{S} \\
P_{k,l}(u,v) &\mapsto P_{k,l}(xux,xvx).
\end{aligned}$$

It follows from Lemma 24 that ξ_x is well-defined.

Furthermore, $\Delta_x^{k+l}\xi_x : \mathcal{S}_{k,l} \to \mathcal{H}_{k,l} \otimes \mathbb{S}$ is a Spin(m)-invariant map; this follows from

$$
\begin{aligned}
\Delta_x^{k+l}\xi_x L(s) P_{k,l}(u,v) &= \Delta_x^{k+l}\xi_x s P_{k,l}(\bar{s}us, \bar{s}vs) \\
&= \Delta_x^{k+l} s P_{k,l}(\bar{s}xuxs, \bar{s}xvxs) \\
&= \Delta_x^{k+l} s P_{k,l}(\bar{s}xs\bar{s}u\bar{s}sxs, \bar{s}x\bar{s}vs\bar{s}xs) \\
&= \Delta_x^{k+l} L_Q(s) P_{k,l}(xux, xvx) \\
&= L_Q(s)\Delta_x^{k+l}\xi_x P_{k,l}(u,v) \qquad (7.14)
\end{aligned}
$$

which is visualised in

$$
\begin{tikzcd}
\mathcal{S}_{k,l} \arrow[r, "\Delta_x^{k+l}\xi_x"] \arrow[d, "L(s)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{S}_{k,l} \arrow[d, "L_Q(s)"] \\
\mathcal{S}_{k,l} \arrow[r, "\Delta_x^{k+l}\xi_x"'] & \mathcal{S}_{k,l}
\end{tikzcd}
$$

Schur's lemma

The image of the Spin(m)-invariant map $\Delta_x^{k+l}\xi_x : \mathcal{S}_{k,l} \to \mathcal{H}_{k,l} \otimes \mathbb{S}$ equals $\mathcal{S}_{k,l}$, since $\mathcal{H}_{k,l} \otimes \mathbb{S} = \mathcal{S}_{k,l} \oplus \widetilde{v}\mathcal{S}_{k,l-1} \oplus u\mathcal{S}_{k-1,l} \oplus \langle \widetilde{u,v} \rangle \mathcal{S}_{k-1,l-1}$. Schur's lemma (see e.g. Lemma 2) implies that there exists a constant C such that

$$
\Delta_x^{k+l} P_{k,l}(xux, xvx) = C P_{k,l}(u,v). \qquad (7.15)
$$

In order to dermine C, we complexify u, v, $\mathfrak{f}_1$ and $\mathfrak{f}_2$ as follows. If $u = e_1 - ie_2$, $v = e_3 - ie_4$, $\mathfrak{f}_1 = e_1 + ie_2$ and $\mathfrak{f}_2 = e_3 + ie_4$, then $\langle u_i, \mathfrak{f}_j \rangle = 2\delta_{ij}$ and $P_{k,l}(u,v) = 2^{k+l} I_{12}$. Furthermore,

$$
\begin{aligned}
\langle xux, \mathfrak{f}_1 \rangle &= 2(x_3^2 + x_4^2 + \cdots + x_m^2) \\
\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2 \rangle &= 4|x|^2(x_5^2 + x_6^2 + \cdots + x_m^2).
\end{aligned}
$$

If we introduce new notations

$$
\begin{aligned}
x_{(3)} &:= (0, 0, x_3, x_4, \ldots, x_m) \\
x_{(5)} &:= (0, 0, 0, 0, x_5, x_6, \ldots, x_m),
\end{aligned}
$$

the expression for $P_{k,l}(xux, xvx)$ yields

$$\begin{aligned} P_{k,l}(xux, xvx) &= 2^{k+l}|x|^{2l}|x_{(3)}|^{2(k-l)}|x_{(5)}|^{2l} I_{12} \\ &= |x|^{2l}|x_{(3)}|^{2(k-l)}|x_{(5)}|^{2l} P_{k,l}(u,v). \end{aligned}$$

A straightforward calculation leads to

$$\Delta_x^a |x|^{2b} = \sum_{j=0}^{\min(a,b)} \left(\begin{array}{c} a \\ j \end{array} \right) 2^{2j}\, j! \left(\begin{array}{c} b \\ j \end{array} \right) \frac{\Gamma(\frac{m}{2} + \mathbb{E}_x - b + a + j)}{\Gamma(\frac{m}{2} + \mathbb{E}_x - b + a)} |x|^{2(b-j)} \Delta_x^{a-j}. \tag{7.16}$$

When Δ_x^a acts on $P_{k,l}(xux, xvx)$, which is $2(k+l)$-homogeneous in x, only the following terms in its expression (7.16) are relevant:

$$\Delta_x^a |x|^{2b} = 2^{2b} \frac{a!}{(a-b)!} \frac{\Gamma(\frac{m}{2} + a)}{\Gamma(\frac{m}{2} + a - b)} \Delta_x^{a-b},$$

for $a \geq b$, and in particular for $a = b$:

$$\Delta_x^a |x|^{2a} = 2^{2a} a! \frac{\Gamma(\frac{m}{2} + a)}{\Gamma(\frac{m}{2})}.$$

This also can be computed directly using polar coordinates and the polar decomposition of Δ_x (see [70]).

By means of the previous results, we find

$$\begin{aligned} &\Delta_x^{k+l} P_{k,l}(xux, xvx) \\ &= \Delta_x^{k+l} |x|^{2l}|x_{(3)}|^{2(k-l)}|x_{(5)}|^{2l} P_{k,l}(u,v) \\ &= 2^{2l} \frac{(k+l)!}{k!} \frac{\Gamma(\frac{m}{2} + k + l)}{\Gamma(\frac{m}{2} + k)} \Delta_{x_{(3)}}^k |x_{(3)}|^{2(k-l)}|x_{(5)}|^{2l} P_{k,l}(u,v) \\ &= 2^{2l} \frac{(k+l)!}{k!} \frac{\Gamma(\frac{m}{2} + k + l)}{\Gamma(\frac{m}{2} + l)} 2^{2(k-l)} \frac{k!}{l!} \frac{\Gamma(\frac{m-2}{2} + k)}{\Gamma(\frac{m-2}{2} + l)} \Delta_{x_{(5)}}^l |x_{(5)}|^{2l} P_{k,l}(u,v) \\ &= 2^{2(k+l)} (k+l)! \frac{\Gamma(\frac{m}{2} + k + l)}{\Gamma(\frac{m}{2} + k)} \frac{\Gamma(\frac{m-2}{2} + k)}{\Gamma(\frac{m-2}{2} + l)} \frac{\Gamma(\frac{m-4}{2} + l)}{\Gamma(\frac{m-4}{2})} P_{k,l}(u,v). \end{aligned}$$

It thus follows from (7.15) that

$$C = 2^{2(k+l)} (k+l)! \frac{\Gamma(\frac{m}{2} + k + l)}{\Gamma(\frac{m}{2} + k)} \frac{\Gamma(\frac{m-2}{2} + k)}{\Gamma(\frac{m-2}{2} + l)} \frac{\Gamma(\frac{m-4}{2} + l)}{\Gamma(\frac{m-4}{2})}. \tag{7.17}$$

Since

$$\begin{aligned}\Delta_x^{k+l}\xi_x L(s)P_{k,l}(u,v) &= L_Q(s)\Delta_x^{k+l}P_{k,l}(xux,xvx)\\ &= CL_Q(s)P_{k,l}(u,v) = CL(s)P_{k,l}(u,v),\end{aligned}$$

the identity (7.15) holds for each $P_{k,l}(u,v)$ in $\mathcal{S}_{k,l}$, by irreducibility.

Finally, using (7.15) and (7.17), the right-hand side of (7.13) leads to

$$\begin{aligned}&\mathcal{Q}_{k,l}\left(|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx)\right)\\ &= -\frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m-4}{2})}\frac{\Gamma(\frac{m-2}{2}+k)}{\Gamma(\frac{m}{2}+k)}\frac{\Gamma(\frac{m-4}{2}+l)}{\Gamma(\frac{m-2}{2}+l)}P_{k,l}(u,v)\delta(x)\\ &= -\frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}\frac{(\frac{m}{2}-2)(\frac{m}{2}-1)}{(\frac{m}{2}+k-1)(\frac{m}{2}+l-2)}P_{k,l}(u,v)\delta(x)\\ &= -A_m\frac{(m-2)(m-4)}{(m+2k-2)(m+2l-4)}P_{k,l}(u,v)\delta(x)\\ &=: (C_{k,l})^{-1}\delta(x)P_{k,l}(u,v).\end{aligned}$$

This defines the constant $C_{k,l}$ that was introduced in Lemma 23.

Conclusion

To recapitulate the results of this section, we state the following theorem.

Theorem 24. *If $C_{k,l}$ is the constant given by*

$$C_{k,l} = -\frac{1}{A_m}\frac{(m+2k-2)(m+2l-4)}{(m-2)(m-4)}, \tag{7.18}$$

the distribution

$$e_{k,l}(x) = C_{k,l}|x|^{-m+1}L(\frac{x}{|x|}) \in \mathcal{C}^\infty(\mathbb{R}^m\setminus\{0\}, End(\mathcal{S}_{k,l})) \tag{7.19}$$

satisfies, for every $P_{k,l}\in\mathcal{S}_{k,l}$, in distributional sense

$$\mathcal{Q}_{k,l}e_{k,l}(x)P_{k,l}(u,v) = \delta(x)P_{k,l}(u,v). \tag{7.20}$$

7.3.3 Reproducing kernel

The relation (7.20) depends on $P_{k,l} \in \mathcal{S}_{k,l}$. For this reason we introduce the reproducing kernel $K_{k,l}$ satisfying

$$(K_{k,l}(u,v;u',v'), P_{k,l}(u,v))_{(u,v)} = P_{k,l}(u',v'),$$

with $(\cdot,\cdot)_{(u,v)}$ the Fischer inner product (6.4) on $\mathcal{S}_{k,l}$.

Theorem 25. *For $e_{k,l}$ defined in (7.19), we have that*

$$E_{k,l}(x;u,v;u',v') = e_{k,l}(x)K_{k,l}(u,v;u',v')$$

satisfies, in distributional sense,

$$\mathcal{Q}_{k,l}E_{k,l}(x;u,v;u',v') = \delta(x)K_{k,l}(u,v;u',v'). \tag{7.21}$$

Proof. With

$$\mathcal{Q}_{k,l} = \left(\mathbf{1} + \frac{u'\partial_{u'}}{m+2k-2} + \frac{v'\partial_{v'}}{m+2l-4} - 2\frac{u'\langle v',\partial_{u'}\rangle\partial_{v'}}{(m+2k-2)(m+2l-4)}\right)\partial_x,$$

we have that

$$\begin{aligned}\big(\mathcal{Q}_{k,l}E_{k,l}(x;u,v;u',v'), P(u,v)\big)_{(u,v)} &= \big(\mathcal{Q}_{k,l}e_{k,l}(x)K_{k,l}(u,v;u',v'), P(u,v)\big)_{(u,v)}\\ &= \mathcal{Q}_{k,l}e_{k,l}(x)\big(K_{k,l}(u,v;u',v'), P(u,v)\big)_{(u,v)}\\ &= \mathcal{Q}_{k,l}e_{k,l}(x)P(u',v')\\ &= \delta(x)P(u',v'). \qquad \square\end{aligned}$$

7.4 Basic integral formulae

The basic integral formulae related to the operators $\mathcal{Q}_{k,l}$ can easily be deduced from Stokes's theorem for the Dirac operator, see e.g. [4] and section 3.2.6.

We introduce the volume element $dx = dx_1 \wedge \ldots \wedge dx_m$ and the oriented surface element $d\sigma_x = \sum_{j=1}^m (-1)^{j-1}e_j d\widehat{x}_j$ with $d\widehat{x}_j = dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$.

Theorem 26 (See also [18]). *Let $\Omega' \subset \mathbb{R}^m$ and $\overline{\Omega} \subset \Omega'$ and let $f, g \in \mathcal{C}^1(\Omega', \mathcal{S}_{k,l})$.*

(i) Stokes's Theorem. One has

$$\int_\Omega \Big[-\big(\mathcal{Q}_{k,l} g(x), f(x)\big)_{(u,v)} + \big(g(x), \mathcal{Q}_{k,l} f(x)\big)_{(u,v)} \Big] dx = \int_{\partial\Omega} \big(g(x), \pi_1(d\sigma_x) f(x)\big)_{(u,v)}.$$

(ii) Cauchy-Pompeiu Theorem. One has

$$- \int_\Omega e_{k,l}(x-y) \mathcal{Q}_{k,l} f(x) dx + \int_{\partial\Omega} e_{k,l}(x-y) \pi_1(d\sigma_x) f(x) = \begin{cases} f(y) & \forall y \in \Omega \\ 0 & \forall y \in \Omega' \setminus \overline{\Omega} \end{cases}$$

(iii) Cauchy integral formula. If $\mathcal{Q}_{k,l} f = 0$ in Ω', one has

$$\int_{\partial\Omega} e_{k,l}(x-y) \pi_1(d\sigma_x) f(x) = \begin{cases} f(y) & \forall y \in \Omega \\ 0 & \forall y \in \Omega' \setminus \overline{\Omega} \end{cases}$$

Proof. Let $f, g \in \mathcal{C}^1(\Omega', \mathcal{S}_{k,l})$. The classical Stokes's formula for the Dirac operator leads to

$$\int_\Omega \big[-(\overline{\partial_x g(x)}) f(x) + \overline{g(x)} (\partial_x f(x)) \big] dx = \int_{\partial\Omega} \overline{g(x)} d\sigma_x f(x).$$

This identity is still depending on the vector variables u and v. To obtain the generalised Stokes's theorem for the operator $\mathcal{Q}_{k,l}$ it is sufficient to take the Fischer inner product with respect to (u, v):

$$\big(\mathcal{Q}_{k,l} g(x), f(x)\big)_{(u,v)} = \big(\pi_1 \partial_x g(x), f(x)\big)_{(u,v)} = (\partial_x g(x), f(x))_{(u,v)}$$

which follows from the Fischer inner product (6.4) and the fact that $\pi_1 = \mathbf{1} - \widetilde{v}\pi_2 - u\pi_3$, due to (6.12). In the same way, we have

$$\begin{aligned} (g(x), \mathcal{Q}_{k,l} f(x))_{(u,v)} &= \overline{(\mathcal{Q}_{k,l} f(x), g(x))}_{(u,v)} \\ &= \overline{(\partial_x f(x), g(x))}_{(u,v)} \\ &= \big(g(x), \partial_x f(x)\big)_{(u,v)} \end{aligned}$$

and

$$(g(x), \pi_1(d\sigma_x)f(x))_{(u,v)} = (g(x), d\sigma_x f(x))_{(u,v)}.$$

The Cauchy-Pompeiu formula for the operator $\mathcal{Q}_{k,l}$ is obtained by substituting $g(x;u,v) = E_{k,l}(x-y;u,v;u',v')$ for the function g in Stokes's formula (i) and using (7.21):

$$\begin{aligned}
&\int_\Omega \Big[-\big(\delta(x-y)K_{k,l}(u,v;u',v'), f(x;u,v)\big)_{(u,v)} \\
&\qquad + \big(E_{k,l}(x-y;u,v;u',v'), \mathcal{Q}_{k,l}f(x;u,v)\big)_{(u,v)}\Big]dx \\
&= \int_{\partial\Omega} \big(E_{k,l}(x-y;u,v;u',v'), \pi_1(d\sigma_x)f(x;u,v)\big)_{(u,v)} \\
\Leftrightarrow\quad & \int_\Omega \Big[-\delta(x-y)f(x;u',v') + \big(E_{k,l}(x-y;u,v;u',v'), \mathcal{Q}_{k,l}f(x)\big)_{(u,v)}\Big]dx \\
&= \int_{\partial\Omega} \big(E_{k,l}(x-y;u,v;u',v'), \pi_1(d\sigma_x)f(x)\big)_{(u,v)} \\
\Leftrightarrow\quad & -f(y;u',v') + \int_\Omega \big(e_{k,l}(x-y)K_{k,l}(u,v;u',v'), \mathcal{Q}_{k,l}f(x)\big)_{(u,v)}dx \\
&= \int_{\partial\Omega} \big(e_{k,l}(x-y)K_{k,l}(u,v;u',v'), \pi_1(d\sigma_x)f(x)\big)_{(u,v)} \\
\Leftrightarrow\quad & -f(y;u',v') - \int_\Omega e_{k,l}(x-y)\big(K_{k,l}(u,v;u',v'), \mathcal{Q}_{k,l}f(x)\big)_{(u,v)}dx \\
&= -\int_{\partial\Omega} e_{k,l}(x-y)\big(K_{k,l}(u,v;u',v'), \pi_1(d\sigma_x)f(x)\big)_{(u,v)} \\
\Leftrightarrow\quad & f(y;u',v') + \int_\Omega e_{k,l}(x-y)\mathcal{Q}_{k,l}f(x;u',v')dx \\
&= \int_{\partial\Omega} e_{k,l}(x-y)\pi_1(d\sigma_x)f(x;u',v'),
\end{aligned}$$

or, in shorthand,

$$f(y) + \int_\Omega e_{k,l}(x-y)\mathcal{Q}_{k,l}f(x)dx = \int_{\partial\Omega} e_{k,l}(x-y)\pi_1(d\sigma_x)f(x).$$

The Cauchy integral formula then follows immediately. $\square$

Chapter 8

Null solutions of $\mathcal{Q}_{k,l}$

As in any function theory linked to a differential operator, a very important piece of knowledge is the full description of its (homogeneous) polynomial null solutions.

Similar to the Rarita-Schwinger operators and the higher spin Dirac operators acting on spinor-valued forms, the spaces of homogeneous polynomial null solutions of $\mathcal{Q}_{k,l}$ will no longer be irreducible as a $\mathrm{Spin}(m)$-module. In section 8.1 we discuss two types (denoted by A and B, respectively) of homogeneous polynomial null solutions for this higher spin operator. These considerations lead in section 8.2 to the formulation of a theorem about the decomposition of $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ into $\mathrm{Spin}(m)$-irreducibles. In the sections 8.3 – 8.7 this theorem is proved by induction on k and l. We end this chapter with some examples.

8.1 Two types of null solutions for $\mathcal{Q}_{k,l}$

A h-homogeneous polynomial f with values in $\mathcal{S}_{k,l}$ belongs to $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ if and only if it satisfies $\pi_1(\partial_x f) = 0$. Similar to the Rarita-Schwinger case and the operator in chapter 5, there are two possibilities to realise this condition. This gives rise to two types of homogeneous polynomial null solutions f for $\mathcal{Q}_{k,l}$: either $\partial_x f = 0$ (called type A solutions) or $\partial_x f \neq 0$ and $\pi_1(\partial_x f) = 0$ (called type B solutions). We will now treat each of these possibilities in detail.

Remark 27. *In what follows, the integers satisfy $h \geq k+l$. We refer to section 2.4.3 for details on the degenerate case $h < k+l$.*

8.1.1 Solutions of type A

As defined in (2.49), the vector space

$$\mathcal{M}^s_{h,k,l} = \{f \in \mathcal{P}_{h,k,l}(\mathbb{R}^m, \mathbb{S}) \,|\, \partial_x f = \partial_u f = \partial_v f = \langle u, \partial_v \rangle f = 0\}$$

is, by construction, precisely the space of h-homogeneous solutions for $\mathcal{Q}_{k,l}$ of type A. Recall from (2.57) the decomposition of this space into irreducibles for Spin(m):

$$\mathcal{M}^s_{h,k,l} = \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{k-l} \langle u, \partial_x \rangle^i \langle \widetilde{v, \partial_x} \rangle^j \mathcal{S}_{h+i+j,k-i,l-j}.$$

To lighten the notation, we will often omit these commuting embedding factors and denote the irreducible modules in the decomposition of $\mathcal{M}^s_{h,k,l}$ by their highest weights only:

$$\mathcal{M}^s_{h,k,l} \cong \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l} (h+i+j, k-i, l-j)'. \tag{8.1}$$

Remark 28. *A necessary condition for a module $\mathcal{S}_{p,q,r}$ to occur, up to an isomorphic copy, in the decomposition of $\mathcal{M}^s_{h,k,l}$, is $p+q+r = h+k+l$.*

8.1.2 Solutions of type B

We have from (6.16) the equivalence

$$\begin{aligned} \mathcal{Q}_{k,l} f = 0 \Leftrightarrow \partial_x f = &\frac{2}{(m+2k-2)(k-l+1)} u \langle \widetilde{\partial_u, \partial_x} \rangle f \\ &+ \frac{2}{(m+2l-4)(k-l+1)} \widetilde{v} \langle \partial_v, \partial_x \rangle f. \end{aligned} \tag{8.2}$$

In view of Proposition 18, the following implication holds:

$$f \in \mathrm{Ker}_h \mathcal{Q}_{k,l} \Rightarrow \partial_x f = u g_1 + \widetilde{v} g_2$$

with $g_1 \in \mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l}$ and $g_2 \in \mathrm{Ker}_{h-1} \mathcal{Q}_{k,l-1}$. Conversely, every $\mathcal{S}_{k,l}$-valued polynomial f of homogeneity degree h in x that satisfies

$$\partial_x f = u g_1 + \widetilde{v} g_2 \tag{8.3}$$

with g_1 and g_2 in $\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l}$ and $\mathrm{Ker}_{h-1} \mathcal{Q}_{k,l-1}$, respectively, also belongs to $\mathrm{Ker}_h \mathcal{Q}_{k,l}$. Now we would like to investigate whether for any choice of these polynomials g_1 and g_2, there exists a polynomial f satisfying (8.3). In other words, we are trying to characterise the conditions which have to be imposed on $g_1 \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l}$ and $g_2 \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}$, such that the following equivalence holds:

$$f \in \mathrm{Ker}_h\mathcal{Q}_{k,l} \quad \Leftrightarrow \quad \begin{cases} \partial_x f = ug_1 + \widetilde{v}g_2 \\ \partial_u f = 0 \\ \partial_v f = 0 \\ \langle u, \partial_v \rangle f = 0 \end{cases}$$

Just like for the Rarita-Schwinger case, this requires the study of compatibility conditions for an inhomogeneous system of equations involving three Dirac operators.

Remark 29. *In [59] the system of three Dirac operators was only proved for the following stable range of m: $m \geq 5$.*

The system above is not of the form considered in [24] due to the presence of the last equation. We will split this system into a simplified system and an extended system. The simplified system is given by

$$\begin{cases} \partial_x f = ug_1 + \widetilde{v}g_2 \\ \partial_u f = 0 \\ \partial_v f = 0 \end{cases} \tag{8.4}$$

and the extended system is defined by adding the condition $\langle u, \partial_v \rangle f = 0$:

$$\begin{cases} \partial_x f = ug_1 + \widetilde{v}g_2 \\ \partial_u f = 0 \\ \partial_v f = 0 \\ \langle u, \partial_v \rangle f = 0 \end{cases} \tag{8.5}$$

The next proposition states that it is sufficient to study solutions for (8.4):

Proposition 19. *If $f \in \mathcal{P}_{h,k,l}(\mathbb{R}^{3m}, \mathbb{S})$ is a solution of (8.4), then the projection $\Pi(f)$ of f on the kernel of $\langle u, \partial_v \rangle$ satifies (8.5):*

$$\begin{cases} \partial_x \Pi(f) = ug_1 + \widetilde{v}g_2 \\ \partial_u \Pi(f) = 0 \\ \partial_v \Pi(f) = 0 \\ \langle u, \partial_v \rangle \Pi(f) = 0 \end{cases}$$

Proof. Using the Fischer decomposition with respect to the operator $\langle u, \partial_v \rangle$, we can write any solution f of (8.4) as

$$f \;=\; f_{k,l} + \langle v, \partial_u \rangle f_{k+1,l-1} + \ldots + \langle v, \partial_u \rangle^l f_{k+l,0} \;=\; \sum_{j=0}^{l} \langle v, \partial_u \rangle^j f_{k+j,l-j}$$

with $\langle u, \partial_v \rangle f_{k+j,l-j} = 0$ for all $0 \leq j \leq l$. Define the projection map Π by

$$\begin{aligned} \Pi : \mathcal{P}_{h,k,l}(\mathbb{R}^{3m}, \mathbb{S}) \;&\rightarrow\; \mathcal{P}_{h,k,l}(\mathbb{R}^{3m}, \mathbb{S}) \cap \mathrm{Ker}\langle u, \partial_v \rangle \\ f \;&\mapsto\; \Pi(f) = f_{k,l}. \end{aligned}$$

We will prove that $f_{k,l}$ satisfies the system of equations (8.5). Because $\partial_u f = 0$ and $[\partial_u, \langle v, \partial_u \rangle] = 0$, we already have that $\partial_u f_{k+j,l-j} = 0$. Combining this result with $\partial_v f = 0$, which means that also the commutator $[\partial_v, \langle v, \partial_u \rangle] = \partial_u$ acts trivially, we find that $\partial_v f_{k+j,l-j} = 0$ holds too. Finally, we verify that $\partial_x f_{k,l} = u g_1 + \widetilde{v} g_2$. Since $[\partial_x, \langle u, \partial_v \rangle] = 0$, it is easily seen that $\Pi(\partial_x f) = \partial_x \Pi(f)$ and hence

$$\partial_x f_{k,l} = \partial_x \Pi(f) = \Pi(\partial_x f) = \Pi(u g_1 + \widetilde{v} g_2) = u g_1 + \widetilde{v} g_2$$

because we have by construction that $u g_1 + \widetilde{v} g_2 \in \mathrm{Ker}\langle u, \partial_v \rangle$. Note that $f_{k,l} \neq 0$, since otherwise $u g_1 + \widetilde{v} g_2 = 0$. $\square$

To any inhomogeneous system of Dirac equations of the form

$$\left\{ \begin{array}{l} \partial_x f = h_1 \\ \partial_u f = h_2 \\ \partial_v f = h_3 \end{array} \right.$$

corresponds the following set of compatibility conditions (see [24]):

$$\left\{ \begin{array}{l} \Delta_u h_1 + \partial_x \partial_u h_2 = 0 \\ \Delta_v h_1 + \partial_x \partial_v h_3 = 0 \\ \Delta_v h_2 + \partial_u \partial_v h_3 = 0 \\ \Delta_x h_2 + \partial_u \partial_x h_1 = 0 \\ \Delta_x h_3 + \partial_v \partial_x h_1 = 0 \\ \Delta_u h_3 + \partial_v \partial_u h_2 = 0 \\ \{\partial_x, \partial_u\} h_3 = \partial_v (\partial_x h_2 + \partial_u h_1) \\ \{\partial_u, \partial_v\} h_1 = \partial_x (\partial_u h_3 + \partial_v h_2) \\ \{\partial_v, \partial_x\} h_2 = \partial_u (\partial_v h_1 + \partial_x h_3) \end{array} \right. .$$

The last three relations (which are linear dependent) are the *radial algebra* relations, which were investigated in [58, 59]. In the present case (8.4), we put $h_1 = ug_1 + \widetilde{v}g_2$ and $h_2 = 0 = h_3$. Motivated by the Rarita-Schwinger case, we will split these conditions into two sets. First of all, define the compatibility conditions of type I (denoted CC-I):

$$\text{CC-I} \leftrightarrow \begin{cases} \Delta_u(ug_1 + \widetilde{v}g_2) = 0 \\ \Delta_v(ug_1 + \widetilde{v}g_2) = 0 \\ \partial_u\partial_v(ug_1 + \widetilde{v}g_2) = 0 \\ \partial_v\partial_u(ug_1 + \widetilde{v}g_2) = 0 \\ \{\partial_u, \partial_v\}(ug_1 + \widetilde{v}g_2) = 0 \end{cases}$$

together with the extra condition

$$\langle u, \partial_v \rangle (ug_1 + \widetilde{v}g_2) = 0.$$

We are then left with two compatibility conditions of type II (denoted CC-II):

$$\text{CC-II} \leftrightarrow \begin{cases} \partial_u\partial_x(ug_1 + \widetilde{v}g_2) = 0 & \text{(i)} \\ \partial_v\partial_x(ug_1 + \widetilde{v}g_2) = 0 & \text{(ii)} \end{cases}$$

These conditions are related as follows:

$$\begin{aligned} \partial_u\partial_x(ug_1 + \widetilde{v}g_2) = 0 \;&\Rightarrow\; \langle u, \partial_v \rangle \partial_u\partial_x(ug_1 + \widetilde{v}g_2) = 0 \\ &\Leftrightarrow\; \partial_v\partial_x(ug_1 + \widetilde{v}g_2) = 0. \end{aligned}$$

This means that it is sufficient to check CC-II (i). However, it turns out that it is useful to investigate both conditions.

It was proved in section 4.2.2 (see also [18]) that the analysis of compatibility conditions in the case of the Rarita-Schwinger operator leads to the conclusion that the kernel space for $\mathcal{R}_{k-1}$ can be embedded into the kernel space for $\mathcal{R}_k$. The compatibility conditions exactly determine the type B solutions, and thus the structure of the kernel space.

In the present case of the operator $\mathcal{Q}_{k,l}$, it is not difficult to show that the conditions of CC-I are equivalent with g_1 and g_2 being elements of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k-1,l})$ and $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l-1})$, respectively. In other words, these conditions determine the values of g_1 and g_2, similarly to the case of $\mathcal{R}_k$. The conditions of the system CC-II however are not equivalent with *every* $g_1 \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l}$ and *every* $g_2 \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}$.

Demanding that CC-II (ii) is satisfied, leads to

$$\begin{aligned}
\partial_v \partial_x (u g_1) &= -\,\partial_v \partial_x \left(\widetilde{v} g_2 \right) \\
\Leftrightarrow\quad -2u \langle \partial_v, \partial_x \rangle g_1 &= -\,(k-l+1)(m+2l-6)\partial_x g_2 \\
&\quad + 2\left(\frac{k-l+1}{k-l+2} \right) \widetilde{v} \langle \partial_v, \partial_x \rangle g_2 - \frac{2}{k-l+2} u \langle \widetilde{\partial_u, \partial_x} \rangle g_2 \\
\Leftrightarrow\quad -2u \langle \partial_v, \partial_x \rangle g_1 &= -\,2\left(\frac{m+2l-4}{m+2k-2} \right) u \langle \widetilde{\partial_u, \partial_x} \rangle g_2 \\
\Leftrightarrow\quad \langle \partial_v, \partial_x \rangle g_1 &= \left(\frac{m+2l-4}{m+2k-2} \right) \langle \widetilde{\partial_u, \partial_x} \rangle g_2,
\end{aligned}$$

where we have used the following fact:

$$\mathcal{Q}_{k,l-1} g_2 = 0 \;\Leftrightarrow\; \partial_x g_2 = \frac{2u \langle \widetilde{\partial_u, \partial_x} \rangle g_2}{(k-l+2)(m+2k-2)} + \frac{2\widetilde{v} \langle \partial_v, \partial_x \rangle g_2}{(k-l+2)(m+2l-6)}.$$

All this means that there exists a polynomial f in $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ such that $\partial_x f = u g_1 + \widetilde{v} g_2$, on condition that the polynomials g_1 and g_2 satisfy

$$\langle \partial_v, \partial_x \rangle g_1 = \left(\frac{m+2l-4}{m+2k-2} \right) \langle \widetilde{\partial_u, \partial_x} \rangle g_2. \tag{8.6}$$

In particular, this relation is satisfied for g_1 and g_2 in the kernel of $\langle \partial_v, \partial_x \rangle$ and $\langle \widetilde{\partial_u, \partial_x} \rangle$, respectively. By Proposition 18, both the left- and right-hand side of (8.6) are polynomials in $\mathrm{Ker}_{h-2} \mathcal{Q}_{k-1,l-1}$.

For g_1 and g_2 satisfying this relation, further calculations then show that CC-II (i) holds as well, i.e.

$$\partial_u \partial_x \big(u g_1 + \widetilde{v} g_2 \big) = 0.$$

Summarising, the type B solutions of the operator $\mathcal{Q}_{k,l}$ can be of the following type:

(i) choosing $g_2 = 0$ in (8.6), we have that $\partial_x f = u g_1$ with $g_1 \in \mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l}$ satisfying $\langle \partial_v, \partial_x \rangle g_1 = 0$, hence

$$\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l} \cap \mathrm{Ker}\, \langle \partial_v, \partial_x \rangle \hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{k,l};$$

(ii) choosing $g_1 = 0$ in (8.6), we have that $\partial_x f = \widetilde{v} g_2$ with $g_2 \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}$ satisfying $\widetilde{\langle \partial_u, \partial_x \rangle} g_2 = 0$, hence

$$\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \cap \mathrm{Ker}\,\widetilde{\langle \partial_u, \partial_x \rangle} \hookrightarrow \mathrm{Ker}_h\mathcal{Q}_{k,l};$$

(iii) finally, if $\partial_x f = u g_1 + \widetilde{v} g_2$ where polynomials $g_1, g_2 \neq 0$ in $\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l}$ and $\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}$, respectively, satisfy (8.6), then the following summands could be embedded:

$$\mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\,\widetilde{\langle \partial_u, \partial_x \rangle} \cap \mathrm{Im}\,\langle \partial_v, \partial_x \rangle \hookrightarrow \mathrm{Ker}_h\mathcal{Q}_{k,l}$$

with

$$\begin{aligned} \widetilde{\langle \partial_u, \partial_x \rangle} &: \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \to \mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} \\ \langle \partial_v, \partial_x \rangle &: \mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \to \mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1}. \end{aligned}$$

Because $[\widetilde{\langle \partial_u, \partial_x \rangle}, \langle \partial_v, \partial_x \rangle] = 0$, we can write (iii) as

$$\mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\,\left(\widetilde{\langle \partial_u, \partial_x \rangle}\langle \partial_v, \partial_x \rangle\right) \hookrightarrow \mathrm{Ker}_h\mathcal{Q}_{k,l}$$

where the operators are shown in the following diagram:

$$\begin{tikzcd} \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \arrow[d, "\widetilde{\langle \partial_u, \partial_x \rangle}"'] & \mathrm{Ker}_h\mathcal{Q}_{k,l} \arrow[l, "\langle \partial_v, \partial_x \rangle"'] \arrow[d, "\widetilde{\langle \partial_u, \partial_x \rangle}"] \\ \mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} & \mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \arrow[l, "\langle \partial_v, \partial_x \rangle"] \end{tikzcd}$$

In the special case $k = l$, the twistor operator $\langle \partial_v, \partial_x \rangle$ does not occur; there exists only one twistor operator and (8.6) reduces to

$$\widetilde{\langle \partial_u, \partial_x \rangle} g_2 = 0.$$

The type B solutions of the operator $\mathcal{Q}_{k,k}$ are thus equivalent with polynomials f such that $\partial_x f = \widetilde{v} g_2$ with $g_2 \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k,k-1}$ such that $\widetilde{\langle \partial_u, \partial_x \rangle} g_2 = 0$. In other words, the following summands could be embedded:

$$\mathrm{Ker}_{h-1}\mathcal{Q}_{k,k-1} \cap \mathrm{Ker}\,\widetilde{\langle \partial_u, \partial_x \rangle} \hookrightarrow \mathrm{Ker}_h\mathcal{Q}_{k,k}.$$

Remark 30. *The notation of the vector spaces* $\mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle}$ *and* $\mathrm{Im}\widetilde{\langle \partial_u, \partial_x \rangle}$ *is undetermined because* $\widetilde{\langle \partial_u, \partial_x \rangle}$ *maps solutions for* $\mathcal{Q}_{k,l}$ *to solutions of* $\mathcal{Q}_{k-1,l}$ *for every* $k > l$*; the same holds for* $\langle \partial_v, \partial_x \rangle$*. However we have chosen not to include the domain as to avoid complicated notations; to avoid confusion, we visualise these operators in a diagram that shows the domain and the target space.*

8.2 The decomposition of the kernel space

8.2.1 Formulation of the theorem

At this point, we have obtained an inductive procedure to describe, at least formally, the space of polynomial solutions for the operator $\mathcal{Q}_{k,l}$. Together with the type A solutions, the kernel space of h-homogeneous polynomial null solutions for $\mathcal{Q}_{k,l}$ can be constructed as follows.

Theorem 27. *For* $h \geq k + l$*, the kernel space* $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ *for the invariant first-order operator* $\mathcal{Q}_{k,l}$ *decomposes as*

$$\begin{aligned} &\mathcal{M}^s_{h,k,l} \oplus \left(\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle \right) \\ &\oplus \left(\mathrm{Ker}_{h-1} \mathcal{Q}_{k,l-1} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle} \right) \\ &\oplus \left(\mathrm{Ker}_{h-2} \mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\widetilde{\langle \partial_u, \partial_x \rangle} \cap \mathrm{Im}\langle \partial_v, \partial_x \rangle \right) \hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{k,l}. \end{aligned} \tag{8.7}$$

The operators and kernel spaces in Theorem 27 are visualised in the following diagram:

$$\begin{tikzcd} \mathrm{Ker}_{h-1} \mathcal{Q}_{k,l-1} \arrow[d, "\widetilde{\langle \partial_u, \partial_x \rangle}"'] & & \mathrm{Ker}_h \mathcal{Q}_{k,l} \arrow[ll] \arrow[d] \\ \mathrm{Ker}_{h-2} \mathcal{Q}_{k-1,l-1} & & \mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l} \arrow[ll, "\langle \partial_v, \partial_x \rangle"'] \end{tikzcd}$$

In the special case that $k = l$, we have

Theorem 28. *For* $h \geq k$*, the kernel space* $\mathrm{Ker}_h \mathcal{Q}_{k,k}$ *for the invariant first-order operator* $\mathcal{Q}_{k,k}$ *decomposes as*

$$\mathcal{M}^s_{h,k,k} \oplus \left(\mathrm{Ker}_{h-1} \mathcal{Q}_{k,k-1} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle} \right) \hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{k,k}. \tag{8.8}$$

The operator and kernel spaces in Theorem 28 are visualised in the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}_{h-1}\,\mathcal{Q}_{k,k-1} \arrow[d, "\widetilde{\langle \partial_u, \partial_x\rangle}"'] & & \mathrm{Ker}_h\,\mathcal{Q}_{k,k} \arrow[ll] \\
\mathrm{Ker}_{h-2}\,\mathcal{Q}_{k-1,k-1} & &
\end{tikzcd}$$

A dimensional analysis will show that there is equality (up to an isomorphism) between left- and right-hand side of (8.7) and left- and right-hand side of (8.8). In particular, we prove in the next sections that both Theorem 27 and Theorem 28 translate to

Theorem 29. *For every $h \geq k+l$, the kernel space $\mathrm{Ker}_h\mathcal{Q}_{k,l}$ decomposes as*

$$\begin{aligned}
\mathrm{Ker}_h\,\mathcal{Q}_{k,l} &\cong \bigoplus_{i=0}^{k-l}\bigoplus_{j=0}^{l}\mathcal{M}^s_{h-i-j,k-i,l-j} \\
&\cong \bigoplus_{i=0}^{k-l}\bigoplus_{j=0}^{l}\bigoplus_{p=0}^{k-i-l+j}\bigoplus_{q=0}^{l-j}(h-i-j+p+q,k-i-p,l-j-q)'.
\end{aligned}$$

The result of this theorem is confirmed by our experiments with the computer algebra package for Lie group computations LiE ([50]). The procedure goes as follows. By means of (7.2), one obtains the splitting into irreducible modules of explicit examples of $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$; the tensor products $\mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_{k,l}) \cong \mathcal{P}_h(\mathbb{R}^m)\otimes\mathcal{S}_{k,l}$ are calculated by means of LiE.

Theorem 29 implies that

$$\mathrm{Ker}_h\mathcal{Q}_{k,l} \cong \mathcal{M}^s_{h,k,l} \oplus \left(\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \oplus \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}\right) \setminus \mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} \quad (8.9)$$

which was expected from our earliest LiE experiments.

The decomposition of the kernel space of $\mathcal{Q}_{k,l}$ can be thought of as a 'box' of vector spaces $\mathcal{M}^s_{h-i-j,k-i,l-j}$ of type A solutions, which is visualised in Figure 8.1. The dot on position (i,j) (with $i \geq 0$ on the k-axis and $j \geq 0$ on the l-axis) denotes the vector space $\mathcal{M}^s_{h-k-l+i+j,i,j}$. The degree in homogeneity in x follows from the homogeneity degrees $k-i$ and $l-j$ in u and v, respectively; the integer $h-i-j$ is thus of no importance in the visualisation. Hence the 'box' is determined by the label (k,l) of the upper right corner.

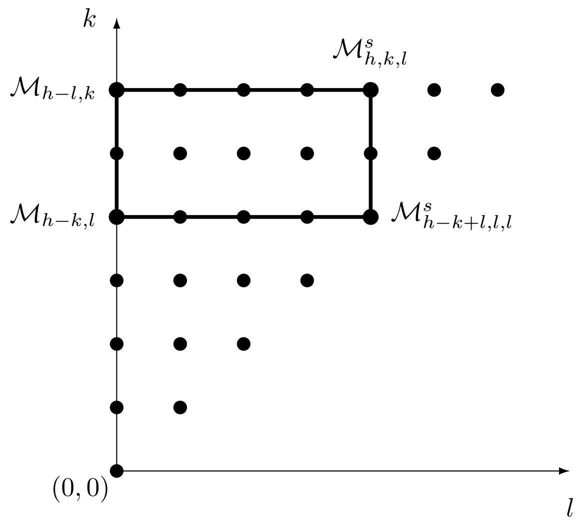


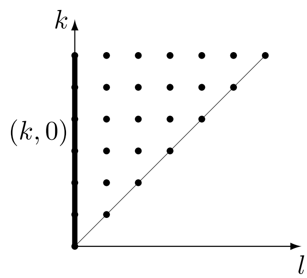
Figure 8.1: The 'box' of Theorem 29.

In [35] it is shown that this remarkable fact may be explained in terms of a non-trivial sequence between certain twistor operators.

8.2.2 Outline of the proof

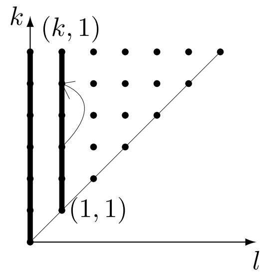
The proof of Theorem 29 goes by induction on the integers k and l. We go through the several steps, which are proved in the next sections.

(i) The case $l = 0$, i.e. the case of the Rarita-Schwinger operator $\mathcal{R}_k$, is briefly recalled in section 8.3.

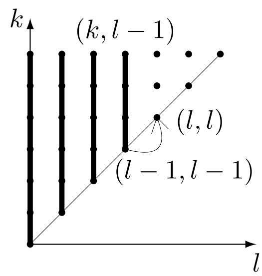


(ii) Section 8.4 deals with the description of $\mathrm{Ker}_h\,\mathcal{Q}_{1,1}$, $\mathrm{Ker}_h\,\mathcal{Q}_{2,1}$ and $\mathrm{Ker}_h\,\mathcal{Q}_{3,1}$.

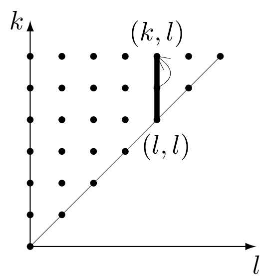
These special cases are an inspiration to establish an induction hypothesis to prove the general case $\mathrm{Ker}_h \mathcal{Q}_{k,1}$ by induction on k.



(iii) We use the results of the previous sections to formulate an induction hypothesis for the structure of $\mathrm{Ker}_h \mathcal{Q}_{k,l-1}, k \geq l-1$ in section 8.5. Using (8.8) together with a dimensional argument, this leads to the construction of $\mathrm{Ker}_h \mathcal{Q}_{l,l}$ in section 8.6.



(iv) Finally, in section 8.7, we decompose $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ into irreducible Spin(m)-modules. The proof goes by induction on k, after having proved the decomposition of $\mathrm{Ker}_h \mathcal{Q}_{l+1,l}$ as the basic step.



Remark 31. *The embedding factors of the Spin(m)-irreducibles in the decomposition of Ker$_h\mathcal{Q}_{k,l}$ are not taken into account and the irreducible modules are systematically denoted by their highest weights only. Furthermore, it follows from Proposition 18 that we have for $k \geq l > 0$ that*

$$\mathrm{Ker}_h\mathcal{Q}_{k,l} \cong \big(\mathrm{Ker}_h\mathcal{Q}_{k,l} \cap \mathrm{Ker}\langle\partial_v,\partial_x\rangle\big) \oplus \big(\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \cap \mathrm{Im}\langle\partial_v,\partial_x\rangle\big) \quad (8.10)$$

and for $k > l \geq 0$ that

$$\mathrm{Ker}_h\mathcal{Q}_{k,l} \cong \big(\mathrm{Ker}_h\mathcal{Q}_{k,l} \cap \mathrm{Ker}\widetilde{\langle\partial_u,\partial_x\rangle}\big) \oplus \big(\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \cap \mathrm{Im}\widetilde{\langle\partial_u,\partial_x\rangle}\big). \quad (8.11)$$

The operators are visualised here:

$$\begin{tikzcd}
\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \arrow[d] & \mathrm{Ker}_h\mathcal{Q}_{k,l} \arrow[l, "\langle\partial_v,\partial_x\rangle"'] \arrow[d, "\widetilde{\langle\partial_u,\partial_x\rangle}"] \\
\mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} & \mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \arrow[l]
\end{tikzcd}$$

Both $\big(\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \cap \mathrm{Im}\langle\partial_v,\partial_x\rangle\big)$ *and* $\big(\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \cap \mathrm{Im}\widetilde{\langle\partial_u,\partial_x\rangle}\big)$ *should be thought of as isomorphic copies inside* $\mathrm{Ker}_h\mathcal{Q}_{k,l}$.

8.3 Decomposition of Ker$_h\mathcal{R}_k$

Recall the decomposition of Ker$_h\mathcal{R}_k$ into irreducible Spin(m)-modules, denoted by their highest weight for convenience:

$$\mathrm{Ker}_h\mathcal{R}_k \cong \bigoplus_{i=0}^{k}\mathcal{M}_{h-i,k-i} \cong \bigoplus_{i=0}^{k}\bigoplus_{j=0}^{k-i}(h-i+j,k-i-j)'. \quad (8.12)$$

The space $\mathcal{M}_{h,k}$ coincides with the h-homogeneous type A solutions for $\mathcal{R}_k$. Note that $\widetilde{\langle\partial_u,\partial_x\rangle} = \langle\partial_u,\partial_x\rangle$ (up to a multiplicative constant) when acting on $\mathcal{R}_k$-valued functions. Hence it follows from (8.11) that

Lemma 25. *For all integers $h \geq k$ one has*

$$\mathrm{Ker}_h\mathcal{R}_k \cap \mathrm{Ker}\langle\partial_u,\partial_x\rangle \cong \mathcal{M}_{h,k}$$

$$\mathrm{Ker}_{h-1}\mathcal{R}_{k-1} \cap \mathrm{Im}\langle\partial_u,\partial_x\rangle \cong \bigoplus_{i=1}^{k}\mathcal{M}_{h-i,k-i} = \bigoplus_{i=0}^{k-1}\mathcal{M}_{h-1-i,k-1-i}.$$

Proof. Using information of chapter 4, this can also be proved explicitly. Note that

$$\mathrm{Ker}_{h-1}\mathcal{R}_{k-1}\cap \mathrm{Im}\langle\partial_u,\partial_x\rangle \cong \bigoplus_{i=0}^{k-1}\mathcal{M}_{h-1-i,k-1-i}\cong \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}.$$

□

8.4 Decomposition of $\mathbf{Ker}_h\mathcal{Q}_{k,1}$

8.4.1 Decomposition of $\mathbf{Ker}_h\mathcal{Q}_{1,1}$

We begin the proof of the decomposition of $\mathrm{Ker}_h\mathcal{Q}_{k,1}$ by taking $k=1$. Hence, the result (5.13) of chapter 5 is useful:

$$\mathrm{Ker}_h\mathcal{Q}_{1,1} \;\cong\; \mathcal{M}^s_{h,1,1}\,\oplus\,\mathcal{M}_{h-1,1} \tag{8.13}$$

which implies that

Lemma 26. *For all integers $h>1$ one has*

$$\begin{aligned}\mathrm{Ker}_h\mathcal{Q}_{1,1}\cap\mathrm{Ker}\langle\partial_v,\partial_x\rangle &\cong \mathcal{M}^s_{h,1,1}\\ \mathrm{Ker}_{h-1}\mathcal{R}_1\cap\mathrm{Im}\langle\partial_v,\partial_x\rangle &\cong \mathcal{M}_{h-1,1}.\end{aligned}$$

Proof. We know from (8.2) that

$$\mathcal{Q}_{1,1}f=0\;\Leftrightarrow\;\partial_x f=\frac{2}{m-2}\widetilde{v}\langle\partial_v,\partial_x\rangle f,$$

proving the first line. The second result then follows from (8.10) and (8.13). □

Note that this is a special case of $\mathrm{Ker}_h\mathcal{Q}_{l,l}$, which is discussed in section 8.6.

8.4.2 Decomposition of $\mathbf{Ker}_h\mathcal{Q}_{2,1}$

By means of Lemma 25, Lemma 26 and Theorem 27 we can prove the following:

Proposition 20. *For $h>2$, the space $\mathrm{Ker}_h\mathcal{Q}_{2,1}$ decomposes as*

$$\mathrm{Ker}_h\mathcal{Q}_{2,1}\;\cong\;\mathcal{M}^s_{h,2,1}\,\oplus\,\mathcal{M}^s_{h-1,1,1}\,\oplus\,\mathcal{M}_{h-1,2}\,\oplus\,\mathcal{M}_{h-2,1}.$$

Proof. A dimensional analysis shows that there are no other irreducibles in $\mathrm{Ker}_h\,\mathcal{Q}_{2,1}$ than the ones in the right-hand side of this decomposition. For the dimension of the left-hand side, the generalisation of the CK-extension (7.1) leads to

$$\dim \mathrm{Ker}_h\,\mathcal{Q}_{2,1} = 2^n \begin{pmatrix} h+2n-1 \\ h \end{pmatrix} \frac{(2n+2)\,2n\,(2n-2)}{3}\,.$$

The dimension of the right-hand side is calculated as the sum of the dimensions of the irreducible $\mathrm{Spin}(m)$-modules:

$$\begin{aligned}&\dim \mathcal{S}_{h,2,1} + \dim \mathcal{S}_{h+1,1,1} + \dim \mathcal{S}_{h+1,2} + \dim \mathcal{S}_{h+2,1} + \dim \mathcal{S}_{h-1,1,1}\\ +\ &2\dim \mathcal{S}_{h,1} + \dim \mathcal{S}_{h-1,2} + \dim \mathcal{M}_{h+1} + \dim \mathcal{S}_{h-2,1} + \dim \mathcal{M}_{h-1}.\end{aligned}$$

Using Maple together with the formulas (2.45) and (2.41), it is shown that this sum indeed equals $\dim \mathrm{Ker}_h\,\mathcal{Q}_{2,1}$. □

Lemma 27. *For every integer $h > 2$ one has*

$$\begin{aligned}\mathrm{Ker}_h\,\mathcal{Q}_{2,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x\rangle &\cong \mathcal{M}^s_{h,2,1} \oplus \mathcal{M}^s_{h-1,1,1}\\ \mathrm{Ker}_{h-1}\mathcal{R}_2 \cap \mathrm{Im}\langle \partial_v, \partial_x\rangle &\cong \mathcal{M}_{h-1,2} \oplus \mathcal{M}_{h-2,1}.\end{aligned}$$

Proof. Recall from (8.2) that

$$\mathcal{Q}_{2,1} f = 0 \ \Leftrightarrow\ \partial_x f = \frac{1}{m+2} u\widetilde{\langle \partial_u, \partial_x\rangle} f + \frac{1}{m-2}\widetilde{v}\langle \partial_v, \partial_x\rangle f$$

with the dual twistor operators visualised in

$$\begin{tikzcd} \mathrm{Ker}_{h-1}\mathcal{R}_2 \arrow[d] & \mathrm{Ker}_h\,\mathcal{Q}_{2,1} \arrow[l, "\langle\partial_v,\partial_x\rangle"'] \arrow[d, "\widetilde{\langle\partial_u,\partial_x\rangle}"] \\ \mathrm{Ker}_{h-2}\mathcal{R}_1 & \mathrm{Ker}_{h-1}\,\mathcal{Q}_{1,1} \arrow[l] \end{tikzcd}$$

For $f \in \mathrm{Ker}_h\,\mathcal{Q}_{2,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x\rangle$, this relation reduces to

$$\partial_x f \ =\ \frac{1}{m+2} u\widetilde{\langle \partial_u, \partial_x\rangle} f,$$

which is definitely fulfilled if $f \in \mathcal{M}^s_{h,2,1}$. In case we have $\partial_x f \neq 0$ on the left-hand side, then $\widetilde{\langle \partial_u, \partial_x\rangle} f \in \mathrm{Ker}_{h-1}\,\mathcal{Q}_{1,1}$ by Proposition 18 and

$$\mathcal{Q}_{1,1}\widetilde{\langle \partial_u, \partial_x\rangle} f = 0 \ \Leftrightarrow\ \partial_x\widetilde{\langle \partial_u, \partial_x\rangle} f = 0 \ \Leftrightarrow\ \widetilde{\langle \partial_u, \partial_x\rangle} f \in \mathcal{M}^s_{h-1,1,1}.$$

Assuming that we can invert this operation, we conclude that

$$\mathcal{M}^s_{h,2,1} \oplus \mathcal{M}^s_{h-1,1,1} \hookrightarrow \mathrm{Ker}_h\,\mathcal{Q}_{2,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x\rangle .$$

Both statements in the lemma then follow from Proposition 20 and (8.10). $\square$

In view of formulating a hypothesis for the induction principle, the result of the previous proposition is rewritten as

$$\mathrm{Ker}_h\,\mathcal{Q}_{2,1} \;\cong\; \bigoplus_{i=0}^{1} \bigoplus_{j=0}^{1} \mathcal{M}^s_{h-i-j,2-i,1-j}. \tag{8.14}$$

8.4.3 Decomposition of $\mathrm{Ker}_h\,\mathcal{Q}_{3,1}$

Taking $k = 3$ and $l = 1$, Theorem 27 leads to

$$\begin{aligned}&\mathcal{M}^s_{h,3,1} \;\oplus\; \Big(\mathrm{Ker}_{h-1}\mathcal{Q}_{2,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x\rangle\Big)\\ &\oplus\; \Big(\mathrm{Ker}_{h-1}\mathcal{R}_3 \cap \mathrm{Ker}\langle \partial_u, \partial_x\rangle\Big)\\ &\oplus\; \Big(\mathrm{Ker}_{h-2}\mathcal{R}_2 \cap \mathrm{Im}\langle \partial_u, \partial_x\rangle\langle \partial_v, \partial_x\rangle\Big) \hookrightarrow \mathrm{Ker}_h\,\mathcal{Q}_{3,1}.\end{aligned}$$

By means of Lemma 25 and Lemma 27, we can prove the following.

Proposition 21. *For $h > 3$, the decomposition of the space $\mathrm{Ker}_h\,\mathcal{Q}_{3,1}$ is given by*

$$\mathrm{Ker}_h\,\mathcal{Q}_{3,1} \;\cong\; \mathcal{M}^s_{h,3,1} \;\oplus\; \bigoplus_{i=0}^{1} \mathcal{M}^s_{h-1-i,2-i,1} \;\oplus\; \mathcal{M}_{h-1,3} \;\oplus\; \bigoplus_{i=1}^{2} \mathcal{M}_{h-1-i,3-i}.$$

Proof. As in the previous proposition, the proof goes by counting dimensions of the left-hand side and the right-hand side independently. $\square$

Again with an eye to formulating an induction hypothesis, the result of Proposition 21 may be rewritten as

$$\mathrm{Ker}_h\,\mathcal{Q}_{3,1} \;\cong\; \bigoplus_{i=0}^{2} \bigoplus_{j=0}^{1} \mathcal{M}^s_{h-i-j,3-i,1-j}. \tag{8.15}$$

8.4.4 Decomposition of $\mathrm{Ker}_h \mathcal{Q}_{k,1}$

The results in (8.13), (8.14) and (8.15) inspire the following induction hypothesis:

$$\begin{aligned}
\mathrm{Ker}_h \mathcal{Q}_{k-1,1} &\cong \bigoplus_{i=0}^{k-2} \bigoplus_{j=0}^{1} \mathcal{M}^s_{h-i-j,k-1-i,1-j} \\
&= \bigoplus_{i=0}^{k-2} \mathcal{M}^s_{h-i,k-1-i,1} \oplus \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-1-i,k-1-i}.
\end{aligned} \tag{8.16}$$

Assuming this hypothesis to hold, we can prove the following lemma, featuring the dual twistor operator visualised below:

$$\begin{tikzcd}
\mathrm{Ker}_{h-1}\mathcal{R}_{k-1} \arrow[d] & \mathrm{Ker}_h \mathcal{Q}_{k-1,1} \arrow[l, "\langle \partial_v, \partial_x \rangle"'] \arrow[d] \\
\mathrm{Ker}_{h-2}\mathcal{R}_{k-2} & \mathrm{Ker}_{h-1} \mathcal{Q}_{k-2,1} \arrow[l]
\end{tikzcd} \tag{8.17}$$

Lemma 28. *For every integer $h \geq k$ one has*

$$\begin{aligned}
\mathrm{Ker}_h \mathcal{Q}_{k-1,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle &\cong \bigoplus_{i=0}^{k-2} \mathcal{M}^s_{h-i,k-1-i,1} \\
\mathrm{Ker}_{h-1}\mathcal{R}_{k-1} \cap \mathrm{Im}\langle \partial_v, \partial_x \rangle &\cong \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-1-i,k-1-i}.
\end{aligned}$$

Proof. The proof is similar to that of Lemma 27. Adapting (8.2) to the present case, leads to

$$\begin{aligned}
\mathcal{Q}_{k-1,1} f = 0 \Leftrightarrow \partial_x f = &\frac{2}{(m+2k-4)(k-1)} u \widetilde{\langle \partial_u, \partial_x \rangle} f \\
&+ \frac{2}{(m-2)(k-1)} \widetilde{v} \langle \partial_v, \partial_x \rangle f.
\end{aligned}$$

For $f \in \mathrm{Ker}_h \mathcal{Q}_{k-1,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle$, this reduces to

$$\partial_x f = \frac{2}{m+2k-4} u \langle \partial_u, \partial_x \rangle f. \tag{8.18}$$

This condition is satisfied if $f \in \mathcal{M}^s_{h,k-1,1}$. If this is not the case and $\partial_x f \neq 0$, then $\langle \partial_u, \partial_x \rangle f \in \mathrm{Ker}_{h-1} \mathcal{Q}_{k-2,1}$. Acting with $\langle \partial_u, \partial_x \rangle$ on both sides of (8.18), leads to

$$\partial_x \langle \partial_u, \partial_x \rangle f = \frac{2}{m+2k-6} u \langle \partial_u, \partial_x \rangle^2 f.$$

This relation holds for every polynomial f satisfying $\partial_x \langle \partial_u, \partial_x \rangle f = 0$, which implies that $\langle \partial_u, \partial_x \rangle f \in \mathcal{M}^s_{h-1,k-2,1}$. In general, acting with $\langle \partial_u, \partial_x \rangle^i$ $(i \geq 0)$ on both sides of (8.18), leads to

$$\partial_x \langle \partial_u, \partial_x \rangle^i f = \frac{2}{m+2k-2i-4} u \langle \partial_u, \partial_x \rangle^{i+1} f.$$

This is true for all $f \in \mathrm{Ker}_{h-i} \mathcal{Q}_{k-1-i,1}$ that satisfy $\partial_x \langle \partial_u, \partial_x \rangle^i f = 0$, which means that $\langle \partial_u, \partial_x \rangle^i f \in \mathcal{M}^s_{h-i,k-1-i,1}$. We can repeat this until $i = k-2$, because $\langle \partial_u, \partial_x \rangle^{k-2} f \in \mathrm{Ker}_{h-k+2} \mathcal{Q}_{1,1}$ and $\langle \partial_u, \partial_x \rangle^{k-1} f \equiv 0$. Combining these results, we have

$$\bigoplus_{i=0}^{k-2} \mathcal{M}^s_{h-i,k-1-i,1} \hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{k-1,1} \cap \mathrm{Ker} \langle \partial_v, \partial_x \rangle.$$

Furthermore, in view of Lemma 25, we have

$$\mathrm{Ker}_{h-1} \mathcal{R}_{k-1} \cap \mathrm{Im} \langle \partial_v, \partial_x \rangle \subset \mathrm{Ker}_{h-1} \mathcal{R}_{k-1} \cong \bigoplus_{i=0}^{k-1} \mathcal{M}_{h-1-i,k-1-i}.$$

Both statements of the present lemma then follow from (8.10) and the induction hypothesis (8.16):

$$\begin{aligned} \mathrm{Ker}_h \mathcal{Q}_{k-1,1} &\cong \bigoplus_{i=0}^{k-2} \mathcal{M}^s_{h-i,k-1-i,1} \oplus \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-1-i,k-1-i} \\ &\cong \Big(\mathrm{Ker}_h \mathcal{Q}_{k-1,1} \cap \mathrm{Ker} \langle \partial_v, \partial_x \rangle \Big) \oplus \Big(\mathrm{Ker}_{h-1} \mathcal{R}_{k-1} \cap \mathrm{Im} \langle \partial_v, \partial_x \rangle \Big). \end{aligned}$$

□

By means of this lemma, we prove the decomposition of the kernel for operators $\mathcal{Q}_{k,1}$ $(k \geq 1)$.

Proposition 22. *For $h > k$, the kernel space* $\mathrm{Ker}_h \mathcal{Q}_{k,1}$ *decomposes as*

$$\mathrm{Ker}_h \mathcal{Q}_{k,1} \cong \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^{1} \mathcal{M}^s_{h-i-j,k-i,1-j}.$$

Proof. We prove this by induction on k. Suppose that the induction hypothesis (8.16) holds and change the summation indices in Lemma 28 as follows:

$$\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle \cong \bigoplus_{i=1}^{k-1} \mathcal{M}^s_{h-i,k-i,1}$$

$$\mathrm{Ker}_{h-2} \mathcal{R}_{k-1} \cap \mathrm{Im}\langle \partial_v, \partial_x \rangle \cong \bigoplus_{i=1}^{k-1} \mathcal{M}_{h-1-i,k-i}.$$

Using (8.17) and Lemma 25, the decomposition on the left-hand side in Theorem 27 leads to

$$\begin{aligned}
&\mathcal{M}^s_{h,k,1} \oplus \Big(\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,1} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle\Big) \oplus \Big(\mathrm{Ker}_{h-1} \mathcal{R}_k \cap \mathrm{Ker}\langle \partial_u, \partial_x \rangle\Big) \\
&\quad \oplus \Big(\mathrm{Ker}_{h-2} \mathcal{R}_{k-1} \cap \mathrm{Im}\langle \partial_u, \partial_x \rangle \langle \partial_v, \partial_x \rangle\Big) \\
&\cong \mathcal{M}^s_{h,k,1} \oplus \bigoplus_{i=1}^{k-1} \mathcal{M}^s_{h-i,k-i,1} \oplus \mathcal{M}_{h-1,k} \oplus \bigoplus_{i=1}^{k-1} \mathcal{M}_{h-1-i,k-i} \\
&= \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^{1} \mathcal{M}^s_{h-i-j,k-i,1-j}.
\end{aligned}$$

To prove that

$$\mathrm{Ker}_h \mathcal{Q}_{k,1} \cong \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^{1} \mathcal{M}^s_{h-i-j,k-i,1-j}$$

we verified, using Maple, that the dimensions of both sides are equal. □

8.5 Induction hypothesis

Now that we have described the kernel space of the operators $\mathcal{R}_k$ (for every integer $k \geq 0$) and $\mathcal{Q}_{k,1}$ (for every integer $k \geq 1$), we formulate the following induction hypothesis on $l-1$:

$$\operatorname{Ker}_h \mathcal{Q}_{k,l-1} \;\cong\; \bigoplus_{i=0}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-i-j,k-i,l-1-j} \tag{8.19}$$

for every $k \geq l-1$. This can be written as

$$\operatorname{Ker}_h \mathcal{Q}_{k,l-1} \;\cong\; \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-j,k,l-1-j} \oplus \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-i-j,k-i,l-1-j}. \tag{8.20}$$

Both summands in the right-hand side of (8.20) have a special meaning. This will be proved in the next lemma, where the operator of interest is visualised in the diagram below:

$$\begin{tikzcd}
\operatorname{Ker}_{h-1} \mathcal{Q}_{k,l-2} \arrow[d] & \operatorname{Ker}_h \mathcal{Q}_{k,l-1} \arrow[l] \arrow[d, "\widetilde{\langle \partial_u, \partial_x \rangle}"] \\
\operatorname{Ker}_{h-2} \mathcal{Q}_{k-1,l-2} & \operatorname{Ker}_{h-1} \mathcal{Q}_{k-1,l-1} \arrow[l]
\end{tikzcd} \tag{8.21}$$

Lemma 29. *For every integer $h \geq k+l$ one has*

$$\operatorname{Ker}_h \mathcal{Q}_{k,l-1} \cap \operatorname{Ker}\widetilde{\langle \partial_u, \partial_x \rangle} \;\cong\; \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-j,k,l-1-j}$$

$$\operatorname{Ker}_{h-1} \mathcal{Q}_{k-1,l-1} \cap \operatorname{Im}\widetilde{\langle \partial_u, \partial_x \rangle} \;\cong\; \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-i-j,k-i,l-1-j}.$$

Proof. Recall from (8.2) that

$$\begin{aligned}
\mathcal{Q}_{k,l-1} f = 0 \;\Leftrightarrow\; \partial_x f = {} & \frac{2}{(m+2k-2)(k-l+2)} u \widetilde{\langle \partial_u, \partial_x \rangle} f \\
& + \frac{2}{(m+2l-6)(k-l+2)} \widetilde{v} \langle \partial_v, \partial_x \rangle f.
\end{aligned}$$

For $f \in \operatorname{Ker}_h \mathcal{Q}_{k,l-1} \cap \operatorname{Ker}\widetilde{\langle \partial_u, \partial_x \rangle}$, this reduces to

$$\partial_x f = \frac{2}{(m+2l-6)(k-l+2)} \widetilde{v} \langle \partial_v, \partial_x \rangle f. \tag{8.22}$$

In the special case that $f \in \mathcal{M}^s_{h,k,l-1}$, this relation is clearly fulfilled. If this is not the case, then $\langle \partial_v, \partial_x \rangle f \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-2}$ and acting with $\langle \partial_v, \partial_x \rangle$ on both sides of (8.22) leads to

$$\partial_x \langle \partial_v, \partial_x \rangle f = \frac{2}{(m+2l-8)(k-l+3)} \widetilde{v} \langle \partial_v, \partial_x \rangle^2 f \tag{8.23}$$

due to

$$\begin{aligned}[\langle \partial_v, \partial_x \rangle, \widetilde{v}] =& (\mathbb{E}_u - \mathbb{E}_v)\partial_x - u\langle \partial_u, \partial_x \rangle - v\langle \partial_v, \partial_x \rangle \\ =& (\mathbb{E}_u - \mathbb{E}_v)\partial_x - \left(u \widetilde{\langle \partial_u, \partial_x \rangle} + \widetilde{v} \langle \partial_v, \partial_x \rangle \right) (\mathbb{E}_u - \mathbb{E}_v + 1)^{-1}.\end{aligned}$$

(Since $\langle \partial_v, \partial_x \rangle f \in \mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-2} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle}$, this result was to be expected.) The relation (8.23) holds for $\langle \partial_v, \partial_x \rangle f \in \mathcal{M}^s_{h-1,k,l-2}$. In general, $\langle \partial_v, \partial_x \rangle^j f \in \mathrm{Ker}_{h-j}\mathcal{Q}_{k,l-1-j} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle}$ (with $j \geq 0$) and

$$\partial_x \langle \partial_v, \partial_x \rangle^j f = \frac{2}{(m+2l-2j-6)(k-l+j+2)} \widetilde{v} \langle \partial_v, \partial_x \rangle^{j+1} f$$

which is true for all $\langle \partial_v, \partial_x \rangle^j f \in \mathcal{M}^s_{h-j,k,l-1-j}$ with $j \leq l-1$. We can conclude from these results that

$$\bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-j,k,l-1-j} \hookrightarrow \mathrm{Ker}_h \mathcal{Q}_{k,l-1} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle}.$$

Next, combining (8.19) and (8.11) into

$$\begin{aligned}\mathrm{Ker}_h \mathcal{Q}_{k,l-1} \cong& \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-j,k,l-1-j} \oplus \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-i-j,k-i,l-1-j} \\ \cong& \left(\mathrm{Ker}_h \mathcal{Q}_{k,l-1} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle} \right) \oplus \left(\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\widetilde{\langle \partial_u, \partial_x \rangle} \right),\end{aligned}$$

the statement then follows from the fact that $\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\widetilde{\langle \partial_u, \partial_x \rangle} \subset \mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l-1}$ with

$$\begin{aligned}\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l-1} \cong& \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-1-i-j,k-1-i,l-1-j} \\ =& \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-i-j,k-i,l-1-j},\end{aligned}$$

according to the induction hypothesis (8.19) made for every integer $k' \geq l-1$, in particular for $k' = k-1 \geq l-1$. $\square$

Let us for a moment recall the whole of our induction argument, which we have visualised in Figure 8.2.

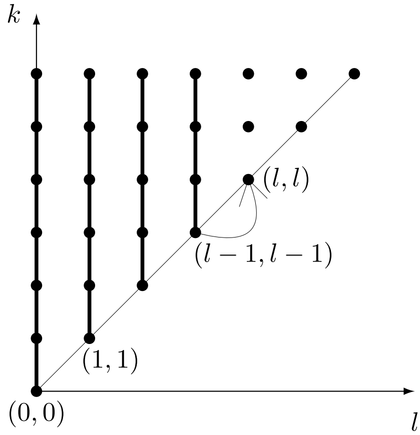


Figure 8.2: Overview of induction principle.

The column $l=0$ represents the known Rarita-Schwinger case $\mathrm{Ker}_h\mathcal{R}_k$. In the column $l=1$ we have established the explicit decomposition of $\mathrm{Ker}_h\mathcal{Q}_{k,1}$ for $k=1,2,3$ and proved it for all integers $k>1$ by induction on k. At this point we assume the decomposition theorem valid for all columns up to and including $l-1$ and all $k \geq l-1$. Then we are able to prove it for the column l: for $k=l$ in section 8.6, and finally for $k>l$ in section 8.7.

8.6 Decomposition of $\mathbf{Ker}_h\mathcal{Q}_{l,l}$

We use the induction hypothesis (8.19) to describe $\mathrm{Ker}_h\mathcal{Q}_{l,l}$.

Proposition 23. *For $h \geq 2l$ one has*

$$\mathrm{Ker}_h\mathcal{Q}_{l,l} \cong \bigoplus_{j=0}^{l} \mathcal{M}^s_{h-j,l,l-j}.$$

Proof. The induction hypothesis (8.19) leads, via Lemma 29 and a translation of summation indices, to

$$\mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle} \;\cong\; \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-j,l,l-j}.$$

The operator $\widetilde{\langle \partial_u, \partial_x \rangle}$ acts as follows:

$$\begin{tikzcd} \mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \arrow[d, "\widetilde{\langle \partial_u, \partial_x \rangle}"'] & & \mathrm{Ker}_h\mathcal{Q}_{l,l} \arrow[ll] \\ \mathrm{Ker}_{h-2}\mathcal{Q}_{l-1,l-1} & & \end{tikzcd}$$

By the left-hand side of the formula in Theorem 28 we obtain

$$\mathcal{M}^s_{h,l,l} \;\oplus\; \left(\mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \cap \mathrm{Ker}\widetilde{\langle \partial_u, \partial_x \rangle} \right) \;=\; \bigoplus_{j=0}^{l} \mathcal{M}^s_{h-j,l,l-j}.$$

Finally, a dimensional analysis of the left- and right-hand side shows that

$$\dim \mathrm{Ker}_h\mathcal{Q}_{l,l} \;=\; \sum_{j=0}^{l} \dim \mathcal{M}^s_{h-j,l,l-j},$$

proving the statement. □

Remark 32. *Every irreducible module occurs just once in this decomposition.*

The proof of the following lemma is similar the proof of Lemma 26 and the operator is visualised in the diagram below:

$$\begin{tikzcd} \mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \arrow[d] & & \mathrm{Ker}_h\mathcal{Q}_{l,l} \arrow[ll, "\langle \partial_v, \partial_x \rangle"'] \\ \mathrm{Ker}_{h-2}\mathcal{Q}_{l-1,l-1} & & \end{tikzcd}$$

Lemma 30. *For all integers $h \geq 2l$ one has*

$$\mathrm{Ker}_h\mathcal{Q}_{l,l} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle \;\cong\; \mathcal{M}^s_{h,l,l}$$

$$\mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \cap \mathrm{Im}\langle \partial_v, \partial_x \rangle \;\cong\; \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-j,l,l-j}.$$

Proof. The first result follows from the definition (6.17):

$$\mathcal{Q}_{l,l}f = 0 \;\Leftrightarrow\; \partial_x f = \frac{2}{m+2l-4}\widetilde{v}\langle\partial_v,\partial_x\rangle f.$$

Indeed, if $\langle\partial_v,\partial_x\rangle f = 0$, then $\partial_x f = 0$, which implies that $f \in \mathcal{M}^s_{h,l,l}$. The second result follows from (8.10) and Proposition 8.29:

$$\mathrm{Ker}_h\mathcal{Q}_{l,l} \cong \mathcal{M}^s_{h,1,1} \oplus \bigoplus_{j=1}^{l}\mathcal{M}^s_{h-j,l,l-j}.$$

□

Remark 33. *Note that*

$$\mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \cap \mathrm{Im}\langle\partial_v,\partial_x\rangle = \mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \cap \mathrm{Ker}\widetilde{\langle\partial_u,\partial_x\rangle}$$

where the operators are given in the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}_{h-1}\mathcal{Q}_{l,l-1} \arrow[d, "\widetilde{\langle\partial_u,\partial_x\rangle}"'] & \mathrm{Ker}_h\mathcal{Q}_{l,l} \arrow[l, "\langle\partial_v,\partial_x\rangle"'] \\
\mathrm{Ker}_{h-2}\mathcal{Q}_{l-1,l-1} &
\end{tikzcd}$$

This comes as no surprise, as it can be shown by means of the twisted de Rham sequence that we are dealing with an exact sequence in this particular case.

8.7 Decomposition of $\mathrm{Ker}_h\mathcal{Q}_{k,l}$

To construct $\mathrm{Ker}_h\mathcal{Q}_{k,l}$, we use the result of the previous section for $\mathrm{Ker}_h\mathcal{Q}_{l,l}$, and proceed by induction on k. That is, we first perform a basic step (by taking $k = l+1$), followed by an inductive step. The first step is thus the construction of $\mathrm{Ker}_h\mathcal{Q}_{l+1,l}$. To this end, consider the operators in the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}_{h-1}\mathcal{Q}_{l+1,l-1} \arrow[d, "\widetilde{\langle\partial_u,\partial_x\rangle}"'] & \mathrm{Ker}_h\mathcal{Q}_{l+1,l} \arrow[l] \arrow[d] \\
\mathrm{Ker}_{h-2}\mathcal{Q}_{l,l-2} & \mathrm{Ker}_{h-1}\mathcal{Q}_{l,l} \arrow[l, "\langle\partial_v,\partial_x\rangle"]
\end{tikzcd}$$

Then, using the decomposition in Theorem 27, Lemma 29 and Lemma 30, we obtain

$$\begin{aligned}
&\mathcal{M}^s_{h,l+1,l} \oplus \left(\mathrm{Ker}_{h-1}\mathcal{Q}_{l,l} \cap \mathrm{Ker}\langle\partial_v,\partial_x\rangle\right) \oplus \left(\mathrm{Ker}_{h-1}\mathcal{Q}_{l+1,l-1} \cap \mathrm{Ker}\langle\widetilde{\partial_u,\partial_x}\rangle\right) \\
&\quad \oplus \left(\mathrm{Ker}_{h-2}\mathcal{Q}_{l,l-1} \cap \mathrm{Im}\langle\widetilde{\partial_u,\partial_x}\rangle \cap \mathrm{Im}\langle\partial_v,\partial_x\rangle\right) \\
&\cong \mathcal{M}^s_{h,l+1,l} \oplus \mathcal{M}^s_{h-1,l,l} \oplus \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-j,l+1,l-j} \oplus \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-1-j,l,l-j} \\
&= \bigoplus_{i=0}^{1}\bigoplus_{j=0}^{l} \mathcal{M}^s_{h-i-j,l+1-i,l-j}.
\end{aligned}$$

A dimensional analysis shows that indeed

$$\mathrm{Ker}_h\mathcal{Q}_{l+1,l} \cong \bigoplus_{i=0}^{1}\bigoplus_{j=0}^{l} \mathcal{M}^s_{h-i-j,l+1-i,l-j}.$$

This allows us to formulate an induction hypothesis on $k-1$:

$$\mathrm{Ker}_h\mathcal{Q}_{k-1,l} \cong \bigoplus_{i=0}^{k-1-l}\bigoplus_{j=0}^{l} \mathcal{M}^s_{h-i-j,k-1-i,l-j} \tag{8.24}$$

which can be written as

$$\mathrm{Ker}_h\mathcal{Q}_{k-1,l} \cong \bigoplus_{i=0}^{k-1-l} \mathcal{M}^s_{h-i,k-1-i,l} \oplus \bigoplus_{i=0}^{k-1-l}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-1-i,l-j}. \tag{8.25}$$

In the next lemma we prove that these summands have a special meaning.

Lemma 31. *For $h \geq k+l$ one has*

$$\begin{aligned}
\mathrm{Ker}_h\mathcal{Q}_{k-1,l} \cap \mathrm{Ker}\langle\partial_v,\partial_x\rangle &\cong \bigoplus_{i=0}^{k-1-l} \mathcal{M}^s_{h-i,k-1-i,l} \\
\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\langle\partial_v,\partial_x\rangle &\cong \bigoplus_{i=0}^{k-1-l}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-1-i,l-j}.
\end{aligned}$$

Proof. The operator in this lemma is visualised in the following diagram:

$$\begin{tikzcd}
\operatorname{Ker}_{h-1}\mathcal{Q}_{k-1,l-1} \arrow[d] & \operatorname{Ker}_h \mathcal{Q}_{k-1,l} \arrow[l, "\langle\partial_v,\partial_x\rangle"'] \arrow[d] \\
\operatorname{Ker}_{h-2}\mathcal{Q}_{k-2,l-1} & \operatorname{Ker}_{h-1}\mathcal{Q}_{k-2,l} \arrow[l]
\end{tikzcd}$$

We know from (8.2) that

$$\begin{aligned}\mathcal{Q}_{k-1,l}f = 0 \Leftrightarrow \partial_x f = {} & \frac{2}{(m+2k-4)(k-l)} u\widetilde{\langle\partial_u,\partial_x\rangle} f \\ & + \frac{2}{(m+2l-4)(k-l)} \widetilde{v}\langle\partial_v,\partial_x\rangle f.\end{aligned}$$

Now, in case $f \in \operatorname{Ker}_h \mathcal{Q}_{k-1,l} \cap \operatorname{Ker}\langle\partial_v,\partial_x\rangle$, this reduces to

$$\partial_x f = \frac{2}{m+2k-4} u\langle\partial_u,\partial_x\rangle f. \tag{8.26}$$

This condition is satisfied if $f \in \mathcal{M}^s_{h,k-1,l}$. If this is not the case and $\partial_x f \neq 0$, then $\langle\partial_u,\partial_x\rangle f \in \operatorname{Ker}_{h-1}\mathcal{Q}_{k-2,l} \cap \operatorname{Ker}\langle\partial_v,\partial_x\rangle$. In general, acting with $\langle\partial_u,\partial_x\rangle^i$ on (8.26), leads to

$$\partial_x \langle\partial_u,\partial_x\rangle^i f = \frac{2}{m+2k-2i-4} u\langle\partial_u,\partial_x\rangle^{i+1} f. \tag{8.27}$$

This holds for every polynomial f that satisfies $\partial_x\langle\partial_u,\partial_x\rangle^i f = 0$, which implies that $\partial_x\langle\partial_u,\partial_x\rangle^i f \in \mathcal{M}^s_{h-i,k-1-i,l}$. We can repeat this until $i = k-l-1$ and $\langle\partial_u,\partial_x\rangle^{k-l-1} f \in \operatorname{Ker}_{h-k+l+1}\mathcal{Q}_{l,l} \cap \operatorname{Ker}\langle\partial_v,\partial_x\rangle$. Hence,

$$\bigoplus_{i=0}^{k-1-l} \mathcal{M}^s_{h-i,k-1-i,l} \hookrightarrow \operatorname{Ker}_h \mathcal{Q}_{k-1,l} \cap \operatorname{Ker}\langle\partial_v,\partial_x\rangle.$$

The proof then follows from (8.10), (8.25) and the fact that, according to the induction hypothesis of section 8.5, we have

$$\begin{aligned}\operatorname{Ker}_{h-1}\mathcal{Q}_{k-1,l-1} \cap \operatorname{Im}\langle\partial_v,\partial_x\rangle \subset {} & \operatorname{Ker}_{h-1}\mathcal{Q}_{k-1,l-1} \\ \cong {} & \bigoplus_{i=0}^{k-l}\bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-1-i-j,k-1-i,l-1-j} \\ = {} & \bigoplus_{i=0}^{k-l}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-1-i,l-j},\end{aligned}$$

where we changed the summation indices in the last line. $\square$

Note that the results of this lemma may be rewritten as

$$\begin{aligned}
\mathrm{Ker}_h \mathcal{Q}_{k-1,l} \cap \mathrm{Ker}\langle \partial_v, \partial_x \rangle &\cong \bigoplus_{i=1}^{k-l} \mathcal{M}^s_{h+1-i,k-i,l} \\
\mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\langle \partial_v, \partial_x \rangle &\cong \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^{l} \mathcal{M}^s_{h+1-i-j,k-i,l-j}.
\end{aligned}$$

If we combine the last decomposition with the result of Lemma 29, we find

$$\mathrm{Ker}_{h-2} \mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\langle \widetilde{\partial_u, \partial_x} \rangle \cap \mathrm{Im}\langle \partial_v, \partial_x \rangle \cong \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j}.$$

The left-hand side of (8.7) in Theorem 27 then leads to

$$\begin{aligned}
&\mathcal{M}^s_{h,k,l} \oplus \bigoplus_{i=1}^{l} \mathcal{M}^s_{h-i,k,l-i} \oplus \bigoplus_{i=1}^{k-l} \mathcal{M}^s_{h-i,k-i,l} \oplus \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j} \\
&= \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j}.
\end{aligned}$$

Hence, finally, the following proposition.

Proposition 24. *For every integers $h \geq k + l$ and $k \geq l$, the kernel space $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ for the invariant first-order operator $\mathcal{Q}_{k,l}$ decomposes as*

$$\begin{aligned}
\mathrm{Ker}_h \mathcal{Q}_{k,l} &\cong \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j} \\
&\cong \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l} \bigoplus_{p=0}^{k-i-l+j} \bigoplus_{q=0}^{l-j} (h-i-j+p+q, k-i-p, l-j-q)'.
\end{aligned}$$

Proof. It suffices to verify that the dimensions of the left-hand side and the right-hand sides are equal, which is done using Maple and the dimension formulas (2.41) and (7.1). $\square$

Remark 34. *The module $\mathcal{S}_{h,1}$ occurs twice in the decomposition of* $\mathrm{Ker}_h\mathcal{Q}_{2,1}$*. On the one hand, it originates from*

$$\mathcal{M}^s_{h-1,1,1} \cong (h-1,1,1)' \oplus (h,1)',$$

and on the other hand, we have

$$\mathcal{M}_{h-1,2} \cong (h-1,2)' \oplus (h,1)' \oplus (h+1)'.$$

Also, in the decomposition of $\mathrm{Ker}_h\mathcal{Q}_{3,1}$*, there are 3 irreducible modules, $\mathcal{S}_{h+1,1}$, $\mathcal{S}_{h-1,1}$ and $\mathcal{S}_{h,2}$, which occur with multiplicity two. In chapter 10 we try to find, in general, which irreducible modules in* $\mathrm{Ker}_h\mathcal{Q}_{k,l}$ *have multiplicity two or higher. We hope to prove that the corresponding embedding maps are linearly independent.*

In order to obtain a compact expression for the dimension of $\mathcal{M}^s_{h,k,l}$, studied in [10] and section 2.4.3, we apply Proposition 24 to $\mathrm{Ker}_h\mathcal{Q}_{k,l}$, $\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l}$, $\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}$ and $\mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1}$, respectively:

$$\begin{aligned}
\mathrm{Ker}_h\mathcal{Q}_{k,l} &\cong \mathcal{M}^s_{h,k,l} \oplus \bigoplus_{i=1}^{l} \mathcal{M}^s_{h-i,k,l-i} \oplus \bigoplus_{i=1}^{k-l} \mathcal{M}^s_{h-i,k-i,l} \\
&\quad \oplus \bigoplus_{i=1}^{k-l}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j} \\
\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} &\cong \bigoplus_{i=0}^{k-1-l}\bigoplus_{j=0}^{l} \mathcal{M}^s_{h-1-i-j,k-1-i,l-j} \\
&= \bigoplus_{i=1}^{k-l} \mathcal{M}^s_{h-i,k-i,l} \oplus \bigoplus_{i=1}^{k-l}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j}, \\
\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} &\cong \bigoplus_{i=0}^{k-l+1}\bigoplus_{j=0}^{l-1} \mathcal{M}^s_{h-1-i-j,k-i,l-1-j} \\
&= \bigoplus_{j=1}^{l} \mathcal{M}^s_{h-j,k,l-j} \oplus \bigoplus_{i=1}^{k-l+1}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j} \\
\mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} &\cong \bigoplus_{i=1}^{k-l+1}\bigoplus_{j=1}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j}.
\end{aligned}$$

This leads to an alternative dimension formula for the vector space $\mathcal{M}^s_{h,k,l}$.

Proposition 25. *The dimension of $\mathcal{M}^s_{h,k,l}$ is given by*

$$\begin{aligned}
&\frac{k(l-1)}{(k-l+1)(2n-1)(2n-2)} \dim \mathcal{M}^s_{h,k,l} \\
&= \binom{h+2n-1}{h}\binom{k+2n-2}{k-1}\binom{l+2n-3}{l-2}\frac{(2n+k+l-1)}{(k+1)l} \\
&\quad - \binom{h+2n-2}{h-1}\binom{k+2n-3}{k-2}\binom{l+2n-4}{l-3} \\
&\qquad \cdot \frac{(2n+k+l-2)}{(k-1)(k+1)(l-2)l}(2kl+2nk+2nl-l-3k) \\
&\quad + \binom{h+2n-3}{h-2}\binom{k+2n-3}{k-2}\binom{l+2n-4}{l-3}\frac{(2n+k+l-3)}{(k-1)(l-2)}.
\end{aligned}$$

Proof. The reasoning followed above, together with the generalised CK-extension (7.1), leads to

$$\begin{aligned}
\dim \mathcal{M}^s_{h,k,l} =\ & \dim \mathrm{Ker}_h \mathcal{Q}_{k,l} - \dim \mathrm{Ker}_{h-1} \mathcal{Q}_{k-1,l} - \dim \mathrm{Ker}_{h-1} \mathcal{Q}_{k,l-1} \\
& + \dim \mathrm{Ker}_{h-2} \mathcal{Q}_{k-1,l-1} \\
=\ & \binom{h+2n-1}{h}\binom{k+2n-2}{k-1}\binom{l+2n-3}{l-2} \\
& \quad \cdot \frac{(2n+k+l-1)(k-l+1)(2n-1)(2n-2)}{(k+1)k(l-1)l} \\
& - \binom{h+2n-2}{h-1}\binom{k+2n-3}{k-2}\binom{l+2n-3}{l-2} \\
& \quad \cdot \frac{(2n+k+l-2)(k-l)(2n-1)(2n-2)}{(k-1)k(l-1)l} \\
& - \binom{h+2n-2}{h-1}\binom{k+2n-2}{k-1}\binom{l+2n-4}{l-3} \\
& \quad \cdot \frac{(2n+k+l-2)(k-l+2)(2n-1)(2n-2)}{(k+1)k(l-2)(l-1)} \\
& + \binom{h+2n-3}{h-2}\binom{k+2n-3}{k-2}\binom{l+2n-4}{l-3} \\
& \quad \cdot \frac{(2n+k+l-3)(k-l+1)(2n-1)(2n-2)}{(k-1)k(l-2)(l-1)}. \qquad (8.28)
\end{aligned}$$

This expression may be simplified by combining the second and third term, which leads to the result stated in the proposition.

Note that, by (2.58), we have

$$\dim \mathcal{M}^s_{h,k,l} = \sum_{i=0}^{k-l} \sum_{j=0}^{l} \dim \mathcal{S}_{h+i+j,k-i,l-j}. \tag{8.29}$$

and, again using Maple, we verified that this equals (8.28). $\square$

8.8 Examples

We conclude this section by illustrating our results with two explicit examples.

The first example shows the decomposition of $\mathrm{Ker}_h \mathcal{Q}_{3,1}$. The Spin($m$)-irreducible modules can be organised as follows:

$$\begin{array}{c}
(h+2,1,1)' \quad (h,1,1)' \quad (h-2,1,1)' \\
(h+1,2,1)' \quad (h-1,2,1)' \\
(h,3,1)'
\end{array}$$

$$\begin{array}{c}
(h+2)' \qquad (h)' \qquad (h-2)' \\
(h+3,1)' \quad \underline{(h+1,1)'} \quad \underline{(h-1,1)'} \quad (h-3,1)' \\
(h+2,2)' \qquad \underline{(h,2)'} \qquad (h-2,2)' \\
(h+1,3)' \quad (h-1,3)'
\end{array}$$

The three underlined modules have multiplicity 2; the remaining modules occur with multiplicity 1.

The second example is the decomposition of $\mathrm{Ker}_h \mathcal{Q}_{4,2}$.

$$\begin{array}{c}
(h+2,2,2)' \quad (h,2,2)' \quad (h-2,2,2)' \\
(h+1,3,2)' \quad (h-1,3,2)' \\
(h,4,2)'
\end{array}$$

$$\begin{array}{c}
(h+2,1,1)' \qquad (h,1,1)' \qquad (h-2,1,1)' \\
(h+3,2,1)' \quad \underline{(h+1,2,1)'} \quad \underline{(h-1,2,1)'} \quad (h-3,2,1)' \\
(h+2,3,1)' \qquad \underline{(h,3,1)'} \qquad (h-2,3,1)' \\
(h+1,4,1)' \quad (h-1,4,1)'
\end{array}$$

$$
\begin{array}{ccccccccc}
 & & (h+2)' & & (h)' & & (h-2)' & & \\
 & (h+3,1)' & & \underline{(h+1,1)'} & & \underline{(h-1,1)'} & & (h-3,1)' & \\
(h+4,2)' & & \underline{(h+2,2)'} & & \underline{\underline{(h,2)'}} & & \underline{(h-2,2)'} & & (h-4,2)' \\
 & (h+3,3)' & & \underline{(h+1,3)'} & & \underline{(h-1,3)'} & & (h-3,3)' & \\
 & & (h+2,4)' & & (h,4)' & & (h-2,4)' & &
\end{array}
$$

The modules that are underlined once, occur twice in the decomposition; the irreducible Spin(m)-module with highest weight $(h,2)'$, doubly underlined, has multiplicity 3. In chapter 9 we explain in general the multiplicities and the geometry of the irreducibles that occur in the kernel space.

Chapter 9

Geometry of $\mathrm{Ker}_h \mathcal{Q}_{k,l}$

At this point, we know the decomposition of the kernel space of the operator $\mathcal{Q}_{k,l}$ into Spin(m)-irreducibles, which we have denoted by their highest weight only. The aim of this chapter is to reveal the geometry of these modules inside the kernel space. It was mentioned before that some modules occur with multiplicity higher than one. However, in order to justify this statement, we still need to prove that the corresponding embedding maps are linearly independent. We will deal with this last issue in chapter 10.

Let $h \geq k+l$ and $k \geq l$. Recall from Proposition 22 the decomposition of h-homogeneous polynomial solutions of $\mathcal{Q}_{k,l}$ into Spin(m)-irreducibles:

$$\begin{aligned} \mathrm{Ker}_h \mathcal{Q}_{k,l} &\cong \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l} \mathcal{M}^s_{h-i-j,k-i,l-j} && (9.1) \\ &\cong \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l} \bigoplus_{p=0}^{k-i-l+j} \bigoplus_{q=0}^{l-j} (h-i-j+p+q, k-i-p, l-j-q)'. && (9.2) \end{aligned}$$

Certain modules in (9.2) occur more than once. With the present form of the summation however, it is not obvious to see *which* modules do and what their multiplicity is. To solve this problem, we rewrite the result of Proposition 29 in such a way that the sum in the right-hand side of (9.2) is more manageable. An overview of this very technical approach is discussed in section 9.1. In section 9.5, we list the modules and their multiplicities. To end this chapter, we present some examples.

Remark 35 (Notation)**.** *The intervals in this chapter are subspaces of positive integers* $\mathbb{N} \cup \{0\}$*. For example,* $[0, a]$ *equals* $\{0, 1, \ldots, a\}$ *and* $]2l - k, l]$ *equals* $\{2l - k + 1, \ldots, l - 1, l\}$.

9.1 Outline of the procedure

9.1.1 The case $k > l$

The first step is to rewrite (9.1) as

$$\mathrm{Ker}_h \mathcal{Q}_{k,l} \cong \bigoplus_{i=0}^{k} \bigoplus_{j=\max(i,l)}^{\min(l+i,k)} \mathcal{M}^s_{h-i,k+l-j,j-i}. \tag{9.3}$$

Both ways of counting the spaces $\mathcal{M}^s_{p,q,r}$ in $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ are visualised in Figure 9.1, where the dot on position (i, j) denotes the vector space $\mathcal{M}^s_{h-k-l+i+j,i,j}$.

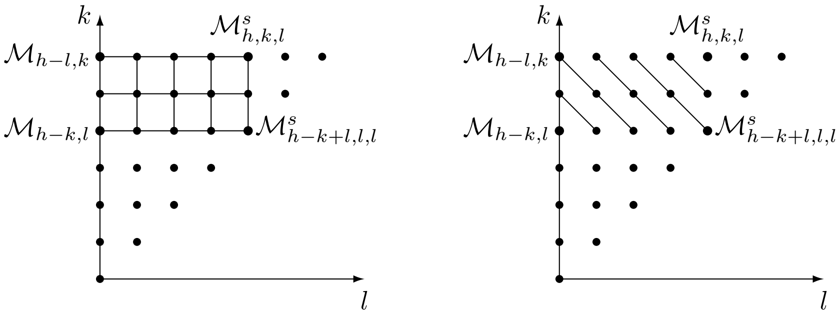


Figure 9.1: Visualisation of (9.1) (left) and (9.3) (right).

Consider the graph on the right in Figure 9.1. Note that $p+q+r = p'+q'+r'$ is a necessary condition for two vector spaces $\mathcal{M}^s_{p,q,r}$ and $\mathcal{M}^s_{p',q',r'}$ to lie on the same diagonal line. This follows from the fact that, on the one hand, the summation index j shifts the vector space $\mathcal{M}^s_{h-i,k+l-j,j-i}$ in (9.3) along the same diagonal line, and that, on the other hand, the sum of indices of this space, given by $(h-i)+(k+l-j)+(j-i) = h+k+l-2i$, is independent of j. Furthermore, recall from Remark 28 that a necessary condition for a module $\mathcal{S}_{a,b,c}$ to be, up to an isomorphic copy, in the decomposition of $\mathcal{M}^s_{p,q,r}$, is $a+b+c=p+q+r$. Hence, to find modules of multiple multiplicities, we only have to consider the diagonal lines in the graph above, which correspond to a fixed index $i \in [0,k]$ in (9.3). This fact explains the alternative summation.

In terms of the highest weights of the Spin(m)-irreducibles, the right-hand side of (9.3) leads to

$$\bigoplus_{i=0}^{k} \bigoplus_{j=\max(i,l)}^{\min(l+i,k)} \bigoplus_{p=0}^{k+l+i-2j} \bigoplus_{q=0}^{j-i} (h-i+p+q, k+l-j-p, j-i-q)' \tag{9.4}$$

which is a complicated summation. Experimenting with LiE showed that a good way of organising these modules is by grouping all modules with the same last weight number. Inspired by this approach, we change the summation indices in (9.4) as follows.

As $0 \leq i \leq k$ and $\max(i,l) \leq j \leq \min(l+i,k)$, the last entry of

$$(h-i+p+q, k+l-j-p, j-i-q)'$$

takes values in $[0,l]$. Now choose $a \in [0,l]$ such that $q = j-i-a \geq 0$. In particular we have $j \geq i+a$, and, since $j \leq \min(l+i,k) \leq k$, we also have that $i+a \leq k$. Changing the summation index $p \to p+j$ in (9.4), this sum can be written as

$$\mathrm{Ker}_h \mathcal{Q}_{k,l} \cong \bigoplus_{a=0}^{l} \bigoplus_{i=0}^{k-a} \bigoplus_{j=\max(i+a,l)}^{\min(l+i,k)} \bigoplus_{p=j}^{k+l+i-j} (h-a-2i+p, k+l-p, a)'. \tag{9.5}$$

Because the weights

$$(h-a-2i+p, k+l-p, a)' \tag{9.6}$$

do not depend on the index j, (9.5) is a very convenient summation to count the multiplicities of modules corresponding to this weight.

In order to get rid of 'max' and 'min' in the summation boundaries for index j in (9.5), we consider three intervals for $i \in [0, k-a]$, due to the boundary cases $i = l-a$ and $i = k-l$. However, if $l-a > k-l \Leftrightarrow a < 2l-k$, the situation is more complicated. It is thus important to make a difference between $k \geq 2l$, discussed in section 9.2, and $k < 2l$. In the latter case, the analysis is a little more involved, because we have to consider $a \in [0, 2l-k]$ and $a \in]2l-k, l]$ separately. These cases are treated in section 9.4 and 9.5, respectively. Finally, examples are given in section 9.7.

9.1.2 The case $k = l$

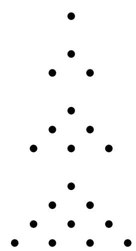
In case $k = l$, both (9.1) and (9.3) lead to

$$\mathrm{Ker}_h\mathcal{Q}_{k,k} \cong \bigoplus_{i=0}^{k}\mathcal{M}^s_{h-i,k,k-i} \cong \bigoplus_{i=0}^{k}\bigoplus_{p=0}^{i}\bigoplus_{q=0}^{k-i}(h-i+p+q, k-p, k-i-q)'.$$

Every module has multiplicity 1 and this sum can be rewritten as

$$\mathrm{Ker}_h\mathcal{Q}_{k,k} \cong \bigoplus_{a=0}^{k}\bigoplus_{p=0}^{k-a}\bigoplus_{i=p}^{k-a}(h-a+p-2i, k-p, a)'$$

which makes it convenient to visualise the modules. For every $a \in [0, k]$, they form a full triangle ($a = k$ leads to one module):



Explicitly, for a fixed in $[0, k]$:

$$\begin{array}{ccccc}
 & & (h-k, a, a)' & & \\
 & (h-k+1, a+1, a)' & & (h-k-1, a, a)' & \\
 & & \vdots & & \\
(h-a, k, a)' & (h-a-2, k, a)' & \cdots & & (h-a-2k, k, a)'
\end{array}$$

Figure 9.2: Visualisation of modules in $\mathrm{Ker}_h\mathcal{Q}_{k,k}$ with $a \in [0, k]$.

For an example, we refer to section 9.7. In what follows, we take $k > l$.

9.2 The case $k \geq 2l$

Let $a \in [0, l]$ be fixed throughout the next sections. The three intervals for i are given by

$$i = \underbrace{0, \ldots, l-a-1}_{\text{interval 1}}, \underbrace{l-a, \ldots, l, \ldots, k-l}_{\text{interval 2}}, \underbrace{k-l+1 \ldots, k-a}_{\text{interval 3}}.$$

Note that $a = l$ automatically leads to only two intervals for i.

9.2.1 The case $i \in [0, l-a-1]$

In this case, $i \leq k-l$ and $i+a \leq l-1$. Hence, the sum (9.5) reduces to

$$\bigoplus_{i=0}^{l-a-1} \bigoplus_{j=l}^{l+i} \bigoplus_{p=j}^{k+l+i-j} (h-a-2i+p, k+l-p, a)'. \tag{9.7}$$

We will now investigate the multiplicities of the modules in (9.7). This goes in two steps. First we take i fixed and give an overview of the modules and their multiplicities. The next step then is to let i run through the interval $[0, l-a-1]$.

The index i is fixed

We take a closer look at the summation over j and p in (9.7). For $j \in [l, l+i]$, the intervals for p are nested as follows:

$$\begin{array}{ccccccccc}
I_{l+i} & & I_{l+i-1} & & & & I_{l+1} & & I_l \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
[l+i, k] & \subset & [l+i-1, k+1] & \subset & \cdots & \subset & [l+1, k+i-1] & \subset & [l, k+i]. \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
j = l+i & & j = l+i-1 & & & & j = l+i-1 & & j = l
\end{array} \tag{9.8}$$

Because the weights in (9.7) do not depend on j, these nested intervals imply there are modules that occur with multiplicity ranging from 1 to $i+1$, i.e. the length of the interval for j. An overview of the modules, classified by their multiplicities:

- Modules of multiplicity $i+1$ are those corresponding to $p \in [l+i, k] = I_{l+i}$, i.e. the smallest interval in (9.8):

$$\bigoplus_{p=l+i}^{k} (h - a - 2i + p, k + l - p, a)'. \tag{9.9}$$

The number of these modules is $k - l - i + 1$.

- There are two modules of multiplicity i: those corresponding to $p = l+i-1$ and $p = k + 1$, which are the endpoints of I_{l+i-1}.

$$\vdots$$

- The modules of multiplicity 2 are the two modules corresponding to $p = l + 1$ and $p = k + i - 1$, which are the endpoints of I_{l+1}.

- Finally, the modules of multiplicity 1 are the two modules corresponding to $p = l$ and $p = k + i$, which are the endpoints of I_l.

Combining, for convenience, the cases of multiplicity $M \in [1, i]$, these are the two modules corresponding to $p = l + M - 1$ and $p = k + i - M + 1$, which are the endpoints of interval I_{l+M-1} in (9.8):

$$\begin{aligned} &(h - a - 2i + l + M - 1, k - M + 1, a)' \\ &\qquad \oplus\ (h - a - i + k - M + 1, l - i + M - 1, a)'. \end{aligned} \tag{9.10}$$

The index i runs through $[0, l - a - 1]$

Next, we classify the modules by their multiplicity if we let i run through $[0, l - a - 1]$. For i fixed, we have multiplicities ranging from 1 to $i + 1$. Hence, for $i \in [0, l - a - 1]$, the range goes from 1 to $l - a$. An overview of the multiplicities and corresponding modules:

- Modules of multiplicity $l - a$ occur only if $i = l - a - 1$. The result in (9.9) leads to $k - 2l + a + 2$ modules:

$$\bigoplus_{p=2l-a-1}^{k} (h - 2l + a + 2 + p, k + l - p, a)'. \tag{9.11}$$

- Modules of multiplicity $l-a-1$ are obtained if i runs through $[l-a-2, l-a-1]$. Indeed, for $i = l-a-2$ the result in (9.9) leads to

$$\bigoplus_{p=2l-a-2}^{k} (h-2l+a+4+p, k+l-p, a)'$$

and if $i = M = l-a-1$, we have from (9.10) that

$$\bigoplus_{i=l-a-1}^{l-a-1} \Big[(h-a-2i+l+M-1, k-M+1, a)' \oplus (h-a-i+k-M+1, l-i+M-1, a)'\Big].$$

- Modules of multiplicity $l-a-2$ are obtained if $i \in [l-a-3, l-a-1]$. Explicitly, for $i = l-a-3$, the result in (9.9) leads to

$$\bigoplus_{p=2l-a-3}^{k} (h-2l+a+6+p, k+l-p, a)'$$

and, because multiplicity $M = l-a-2$ happens only for $i = l-a-2$ and $i = l-a-1$, we have from (9.10) that

$$\bigoplus_{i=l-a-2}^{l-a-1} \Big[(h-a-2i+l+M-1, k-M+1, a)' \oplus (h-a-i+k-M+1, l-i+M-1, a)'\Big].$$

$$\vdots$$

- Finally, modules of multiplicity 1 are obtained for every $i \in [0, l-a-1]$. If $i = 0$, the result in (9.9) leads to

$$\bigoplus_{p=l}^{k}(h-a+p, k+l-p, a)'$$

and if $M = 1$, we have from (9.10) that

$$\bigoplus_{i=1}^{l-a-1} \Big[(h-a-2i+l+M-1, k-M+1, a)' \oplus (h-a-i+k-M+1, l-i+M-1, a)' \Big].$$

The cases of multiplicity $M \in [1, l-a-1]$ can be combined as follows:

$$\bigoplus_{p=l+M-1}^{k} (h-2M+2-a+p, k+l-p, a)' \oplus \bigoplus_{i=M}^{l-a-1} \Big[(h-a-2i+l+M-1, k-M+1, a)' \oplus (h-a-i+k-M+1, l-i+M-1, a)' \Big]. \tag{9.12}$$

This sum represents $k+l+2-2a-3M$ modules of multiplicity $M \in [1, l-a-1]$.

Summary

In the sum (9.7)

- there exist $k-2l+a+2$ modules of multiplicity $l-a$, explicitly given in (9.11);
- there exist $k+l+2-2a-3M$ modules of multiplicity $M \in [1, l-a-1]$, given in (9.12).

Combining these two cases, there exist $k+l+2-2a-3M$ modules of multiplicity $M \in [1, l-a]$ in (9.7).

9.2.2 The case $i \in [l-a, k-l]$

In this case, $i+a \geq l$ and $l+i \leq k$. Hence, for $a \in [0, l]$ fixed, the sum (9.5) reduces to

$$\bigoplus_{i=l-a}^{k-l} \bigoplus_{j=i+a}^{l+i} \bigoplus_{p=j}^{k+l+i-j} (h-a-2i+p, k+l-p, a)'. \tag{9.13}$$

Similarly to the previous case, investigating the multiplicities of the modules in (9.13) goes in two steps.

The index i is fixed

We take a closer look at the summation for j and p in (9.13). For $j \in [i+a, l+i]$, the intervals for p are nested as follows:

$$\begin{array}{ccccccc} I_{l+i} & & I_{l+i-1} & & & & I_{i+a} \\ \downarrow & & \downarrow & & & & \downarrow \\ [l+i,k] & \subset & [l+i-1,k+1] & \subset & \cdots & \subset & [i+a,k+l-a]. \\ \uparrow & & \uparrow & & & & \uparrow \\ j=l+i & & j=l+i-1 & & & & j=i+a \end{array} \tag{9.14}$$

In this case, there are modules that occur with multiplicity ranging from 1 to $l-a+1$, i.e. the length of the interval for j. An overview:

- The modules of multiplicity $l-a+1$ are obtained in case $p \in [l+i, k] = I_{l+i}$, which is the smallest interval in (9.14):

$$\bigoplus_{p=l+i}^{k} (h-a-2i+p, k+l-p, a)'.$$

  There are $k-l-i+1$ modules of multiplicity $l-a+1$.

- The modules of multiplicity $M \in [1, l-a]$ are the two modules corresponding to $p = i+a+M-1$ and $p = k+l-a-M+1$, which are the endpoints of interval $I_{i+a+M-1}$ in (9.14):

$$\begin{aligned} &(h-i+M-1, k+l-i-a-M+1, a)' \\ &\qquad\qquad \oplus \ (h-2i-2a+k+l-M+1, a+M-1, a)'. \end{aligned}$$

The index i runs through $[l-a, k-l]$

As opposed to the previous section, the multiplicities do not depend on i, making this step much easier. Here is an overview of the multiplicities and corresponding modules:

- In total, we have

$$\sum_{i=l-a}^{k-l} (k-l-i+1) = \binom{k-2l+2+a}{2}$$

modules of multiplicity $l-a+1$:

$$\bigoplus_{i=l-a}^{k-l} \bigoplus_{p=l+i}^{k} (h-a-2i+p, k+l-p, a)'. \tag{9.15}$$

- There are $2(k-2l+a+1)$ modules of multiplicity $M \in [1, l-a]$:

$$\bigoplus_{i=l-a}^{k-l} \Big[(h-i+M-1, k+l-i-a-M+1, a)' \\ \oplus \ (h-2i-2a+k+l-M+1, a+M-1, a)' \Big]. \tag{9.16}$$

Summary

In the sum (9.13)

- there exist $\begin{pmatrix} k-2l+2+a \\ 2 \end{pmatrix}$ modules of multiplicity $l-a+1$, given in (9.15);
- there exist $2(k-2l+a+1)$ modules of multiplicity $M \in [1, l-a]$, given in (9.16).

9.2.3 The case $i \in [k-l+1, k-a]$

In this case, $i+a \geq l$ and $l+i > k$, hence for $a \in [0, l]$ fixed, the sum (9.5) reduces to

$$\bigoplus_{i=k-l+1}^{k-a} \bigoplus_{j=i+a}^{k} \bigoplus_{p=j}^{k+l+i-j} (h-a-2i+p, k+l-p, a)'. \tag{9.17}$$

Once again, we will investigate the multiplicities of the modules in (9.17) by considering two steps.

The index i is fixed

Similarly to the previous cases, we take a closer look at the summation of j and p in (9.17). If $j \in [i+a, k]$, then the intervals for p are nested as follows:

$$\begin{array}{ccccccc} I_k & & I_{k-1} & & & & I_{i+a} \\ \downarrow & & \downarrow & & & & \downarrow \\ [k, l+i] & \subset & [k-1, l+i+1] & \subset & \cdots & \subset & [i+a, k+l-a]. \\ \uparrow & & \uparrow & & & & \uparrow \\ j=k & & j=k-1 & & & & j=i+a \end{array} \tag{9.18}$$

In this case, there are modules that occur with multiplicity ranging from 1 to $k-i-a+1$, i.e. the length of the interval for j. Because this is very similar to the case explained in section 9.2.1, we present a compact overview:

- We single out the modules of multiplicity $k-i-a+1$, which are the modules corresponding to $p \in [k, l+i] = I_k$, which is the smallest interval in (9.18):

  $$\bigoplus_{p=k}^{l+i} (h-a-2i+p, k+l-p, a)'. \tag{9.19}$$

  There are $l+i-k+1$ modules of multiplicity $k-i-a+1$.

- The modules of multiplicity $M \in [1, k-i-a]$ are precisely the two modules corresponding to $p = i+a+M-1$ and $p = k+l-a-M+1$, which are the endpoints of interval $I_{i+a+M-1}$ in (9.18):

  $$\begin{aligned} &(h-i+M-1, k+l-i-a-M+1, a)' \\ &\qquad\oplus\ (h-2i-2a+k+l-M+1, a+M-1, a)'. \end{aligned} \tag{9.20}$$

The index i runs in $[k-l+1, k-a]$

The next step now is to classify the modules by their multiplicity if we let i run through $[k-l+1, k-a]$. For i fixed, the multiplicities ranged from 1 to $k-i-a+1$. Hence, for $i \in [k-l+1, k-a]$, the range goes from 1 to $l-a$. An overview:

- Multiplicity $l-a$ occurs only if $i = k-l+1$ and it follows from (9.19) that we have two modules:

$$\bigoplus_{p=k}^{k+1} (h-2k+2l-a-2+p, k+l-p, a)'. \tag{9.21}$$

- Modules of multiplicity $M \in [1, l-a-1]$ are coming from (9.19), in case $i = k-a-M+1$, and from the result in (9.20). This leads to

$$\begin{aligned} &\bigoplus_{p=k}^{l+k-a+1-M} (h-2k+a-2+2M+p, k+l-p, a)' \\ &\oplus \bigoplus_{i=k-l+1}^{k-a-M} \Big[(h-i+M-1, k+l-i-a-M+1, a)' \\ &\qquad \oplus \ (h-2i-2a+k+l-M+1, a+M-1, a)' \Big]. \end{aligned} \tag{9.22}$$

In total, we are dealing with $3l-3a-3M+2$ modules of multiplicity M.

Summary

In the sum (9.17)

- there exist 2 modules of multiplicity $l-a$, explicitly given in (9.21);
- there exist $3l-3a-3M+2$ modules of multiplicity $M \in [1, l-a-1]$, given in (9.22).

In short, in (9.17) there exist $3l-3a-3M+2$ modules of multiplicity $M \in [1, l-a]$.

9.3 Summary of the case $k \geq 2l$

To combine the results from the previous sections, we consider the multiplicities, beginning with the highest one: $l-a+1$. According to (9.15) and after changing the summation indices, these modules are given by

$$\bigoplus_{a=0}^{l} \bigoplus_{j=l}^{k-l+a} \bigoplus_{i=l-a}^{k-j} (h+k+l-a-j-2i, j, a)'.$$

Similar to the Rarita-Schwinger case, modules in this sum can be represented by a full triangle, as shown in Figure 9.3.

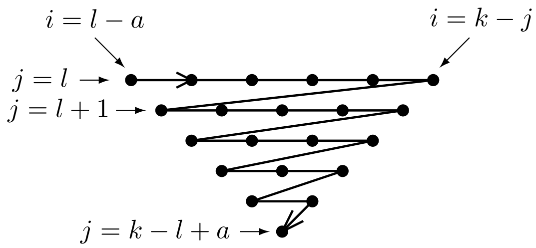


Figure 9.3: Order of counting modules of multiplicity $l - a + 1$ (a is fixed).

Next, we list the modules of multiplicity $l-a$ in (9.5). They are given in (9.11), (9.16) and (9.21):

$$\bigoplus_{a=0}^{l-1}\left(\bigoplus_{p=2l-a-1}^{k}(h-2l+a+2+p,k+l-p,a)'\right.$$
$$\oplus\bigoplus_{i=l-a}^{k-l}\Big[(h-i+l-a-1,k-i+1,a)' \oplus (h-2i-a+k+1,l-1,a)'\Big]$$
$$\left.\oplus\bigoplus_{p=k}^{k+1}(h-2k+2l-a-2+p,k+l-p,a)'\right).$$

Once again, rewriting this sum by changing the summation indices will be more convenient for visualisation. Hence we obtain

$$\bigoplus_{a=0}^{l-1}\bigoplus_{j=l}^{k-l+a+1}\Big[(h+(k-l+a+2-j),j,a)' \oplus (h-(k-l+a+2-j),j,a)'\Big]$$
$$\oplus\bigoplus_{a=0}^{l-1}\bigoplus_{i=l-a}^{k-l+1}(h+k-a+1-2i,l-1,a)'. \tag{9.23}$$

It is not difficult to verify that the modules of multiplicity $l-a$ can be visualised as a hexagon ring of modules surrounding the triangle of modules of multiplicity $l-a+1$. Indeed, the first line in (9.23) gives two modules (for j

fixed) which are symmetric with respect to the value h in the first entry of the weights. The second line gives the top row of modules in Figure 9.4.

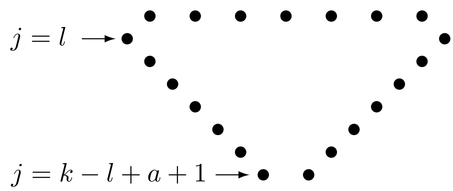


Figure 9.4: Modules of multiplicity $l - a$ (a is fixed).

Finally, we list the modules of multiplicity $M \in [1, l - a - 1]$ that occur in (9.5). We use the results in (9.12), (9.16) and (9.22). Changing the summation indices, this can be written as

$$\begin{aligned}
\bigoplus_{a=0}^{l-2} \bigoplus_{M=1}^{l-a-1} \Bigg(& \bigoplus_{i=l-a}^{k-a+1-M} (h + k + l - 2a + 1 - M - 2i, a - 1 + M, a)' \\
\oplus & \bigoplus_{j=a+M}^{l} \Big[(h + (k - l - a + 2 - 2M + j), j, a)' \\
& \qquad \oplus (h - (k - l - a + 2 - 2M + j), j, a)' \Big] \\
\oplus & \bigoplus_{j=l+1}^{k-M} \Big[(h + (k + l - a + 2 - 2M - j), j, a)' \\
& \qquad \oplus (h - (k + l - a + 2 - 2M - j), j, a)' \Big] \\
\oplus & \bigoplus_{i=M-1}^{l-a} (h + l - a - 1 + M - 2i, k - M + 1, a)' \Bigg) .
\end{aligned} \tag{9.24}$$

These modules (of multiplicity M) can be visualised as rings of hexagons surrounding the previous hexagons (consisting of modules of multiplicity $M + 1$). Indeed, the first (resp. last) line in (9.24) corresponds to the top (resp. bottom) row of modules in Figure 9.5. The second and third line in (9.24) give the rows of the two (symmetrical) modules.

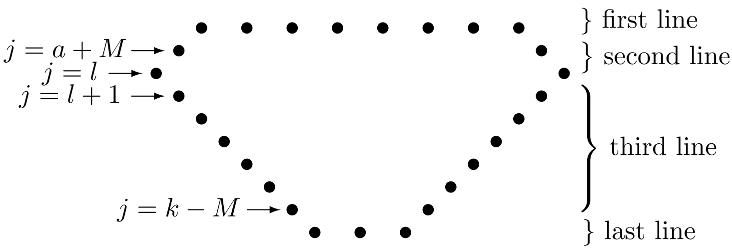


Figure 9.5: Modules of multiplicity $l - a - 1$ (a is fixed).

This geometrical description is confirmed by counting the modules. Indeed, for a fixed, we find

- $\begin{pmatrix} k - 2l + 2 + a \\ 2 \end{pmatrix}$ modules of multiplicity $l - a + 1$, i.e. the full triangle of Figure 9.3;
- $k-2l+a+2+2(k-2l+a+1)+2 = 3(k-2l+2+a)$ modules of multiplicity $l - a$, this corresponds to the most inner hexagon ring, pictured in Figure 9.4;
- $k+l+2-2a-3M+2(k-2l+a+1)+3l-3a-3M+2 = 3(k-a-2M+2)$ modules of multiplicity $M = 1, \dots, l-a-1$. This corresponds to the outer hexagon rings.

Figure 9.6 is obtained by putting all this information together. Note that the multiplicity decreases by one on every outer hexagon ring.

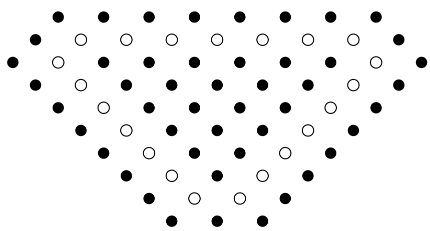


Figure 9.6: Modules for a fixed a.

Remark 36. *Recall from the first chapter that the multiplicities of weights of $\mathfrak{sl}(3, \mathbb{C})$ obey the following pattern. The weights in the outermost hexagon or triangle ring have multiplicity 1. The multiplicities increase by 1 each time one* *moves inward one hexagon until you hit triangles, at which point the multiplicities stabilise. There is thus a remarkable connection between the Spin(m)-irreducible modules in the kernel space for $\mathcal{Q}_{k,l}$ and the weights of $\mathfrak{sl}(3,\mathbb{C})$.*

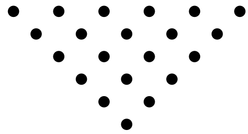
Summarising and letting a run through $[0,l]$, we obtain

$$
\begin{aligned}
\mathrm{Ker}_h \mathcal{Q}_{k,l} \cong\ & \bigoplus_{a=0}^{l} \bigoplus_{j=l}^{k-l+a} \bigoplus_{i=l-a}^{k-j} (l-a+1)(h+k+l-a-j-2i,j,a)' \\
\oplus & \bigoplus_{a=0}^{l-1} \bigoplus_{j=l}^{k-l+a+1} (l-a)\Big[(h+(k-l+a+2-j),j,a)' \\
& \qquad\qquad \oplus\ (h-(k-l+a+2-j),j,a)'\Big] \\
\oplus & \bigoplus_{a=0}^{l-1} \bigoplus_{i=l-a}^{k-l+1} (l-a)(h+k-a+1-2i,l-1,a)' \\
\oplus & \bigoplus_{a=0}^{l-2} \bigoplus_{M=1}^{l-a-1} M \Bigg(\bigoplus_{i=l-a}^{k-a+1-M} (h+k+l-2a+1-M-2i,a-1+M,a)' \\
& \oplus \bigoplus_{j=a+M}^{l} \Big[(h+(k-l-a+2-2M+j),j,a)' \\
& \qquad\qquad \oplus\ (h-(k-l-a+2-2M+j),j,a)'\Big] \\
& \oplus \bigoplus_{j=l+1}^{k-M} \Big[(h+(k+l-a+2-2M-j),j,a)' \\
& \qquad\qquad \oplus\ (h-(k+l-a+2-2M-j),j,a)'\Big] \\
& \oplus \bigoplus_{i=M-1}^{l-a} (h+l-a-1+M-2i,k-M+1,a)' \Bigg).
\end{aligned}
$$

Remark 37. *Singling out the modules with $a=l$, only the following modules occur:*

$$\bigoplus_{j=l}^{k} \bigoplus_{i=0}^{k-j} (h+k-j-2i,j,l)'.$$

They all have multiplicity 1 in $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ *and can be represented by a full triangle of modules:*



9.4 The case $k < 2l$ with $a \in [0, 2l-k]$

If $k - l < l - a$, then we consider three intervals for $i \in [0, k-a]$:

$$i = \underbrace{0, \ldots, k-l-1}_{\text{interval 1}}\,,\ \underbrace{k-l, \ldots, l-a}_{\text{interval 2}}\,,\ \underbrace{l-a+1, \ldots, l, \ldots, k-a}_{\text{interval 3}}.$$

Take $a \in [0, 2l-k]$ fixed.

9.4.1 The case $i \in [0, k-l-1]$

Because $i \leq k-l$ and $i + a \leq l$, the sum (9.5) reduces to

$$\bigoplus_{i=0}^{k-l-1} \bigoplus_{j=l}^{l+i} \bigoplus_{p=j}^{k+l+i-j} (h - a - 2i + p, k + l - p, a)'. \tag{9.25}$$

As before, investigating the multiplicities of the modules in (9.25) goes in two steps: we immediately present the result here.

- In the sum (9.25), there are two modules of multiplicity $k - l$:

$$\bigoplus_{p=k-1}^{k} (h - 2k + 2l - a + 2 + p, k + l - p, a)'. \tag{9.26}$$

- The modules of multiplicity $M \in [1, k-l-1]$ in (9.25) are given by

$$\bigoplus_{p=l+M-1}^{k} (h-2M+2-a+p, k+l-p, a)' \\ \oplus \bigoplus_{i=M}^{k-l-1} \Big[(h-a-2i+l+M-1, k-M+1, a)' \\ \oplus\ (h-a-i+k-M+1, l-i+M-1, a)' \Big]. \qquad (9.27)$$

9.4.2 The case $i \in [k-l, l-a]$

If $i \geq k-l$ and $i+a \leq l$, the sum (9.5) reduces to

$$\bigoplus_{i=k-l}^{l-a} \bigoplus_{j=l}^{k} \bigoplus_{p=j}^{k+l+i-j} (h-a-2i+p, k+l-p, a)'. \qquad (9.28)$$

To find the modules and their multiplicities inside this sum, we proceed as explained in previous sections. We only give the result here.

- In (9.28) there are

$$\sum_{i=k-l}^{l-a} (l-k+1+i) = \begin{pmatrix} 2l-k-a+2 \\ 2 \end{pmatrix}$$

  modules of multiplicity $k-l+1$:

$$\bigoplus_{i=k-l}^{l-a} \bigoplus_{p=k}^{l+i} (h-a-2i+p, k+l-p, a)'. \qquad (9.29)$$

- There exist $2(2l-k-a+1)$ modules of multiplicity $M \in [1, k-l]$ in (9.28):

$$\bigoplus_{i=k-l}^{l-a} \Big[(h-2i-a+l+M-1, k-M+1, a)' \\ \oplus\ (h-i-a+k-M+1, l-i+M-1, a)' \Big]. \qquad (9.30)$$

9.4.3 The case $i \in [l-a+1, k-a]$

Note that only considering a fixed in the smaller interval $[0, l] \subset [0, 2l-k]$ makes sense in this case. Because $i + a \geq l$ and $l + i > k$, the sum (9.5) reduces to

$$\bigoplus_{i=l-a+1}^{k-a} \bigoplus_{j=i+a}^{k} \bigoplus_{p=j}^{k+l+i-j} (h-a-2i+p, k+l-p, a)'. \tag{9.31}$$

We immediately give an overview of the multiplicities of the modules in this sum.

- There are $2l-k-a+2$ modules of multiplicity $k-l$ in (9.31):

$$\bigoplus_{p=k}^{2l-a+1} (h-2l+a-2+p, k+l-p, a)'. \tag{9.32}$$

- In (9.31), there are $2k-l-a-3M+2$ modules of multiplicity $M \in [1, k-l-1]$:

$$\begin{aligned} &\bigoplus_{p=k}^{l+k-a+1-M} (h-2k+a-2+2M+p, k+l-p, a)' \\ &\oplus \bigoplus_{i=l-a+1}^{k-a-M} \Big[(h-i+M-1, k+l-i-a-M+1, a)' \\ &\qquad \oplus (h-2i-2a+k+l-M+1, a+M-1, a)' \Big]. \end{aligned} \tag{9.33}$$

9.5 The case $k < 2l$ with $a \in]2l-k, l]$

There are three intervals to be considered for i ranging from 0 to $k-a$:

$$i = \underbrace{0, \ldots, l-a-1}_{\text{interval 1}}, \underbrace{l-a, \ldots, k-l}_{\text{interval 2}}, \underbrace{k-l+1, \ldots, k-a}_{\text{interval 3}}, k-a+1, \ldots, l.$$

Because this case leads to same results as in section 9.2, we proceed to the summary at once.

9.6 Summary of the case $k<2l$

Using the results of the previous sections and changing the summation indices, we conclude:

$$
\begin{aligned}
\mathrm{Ker}_h\mathcal{Q}_{k,l} \;\cong\; & \bigoplus_{a=0}^{2l-k}\bigoplus_{j=k-l+a}^{l}\bigoplus_{i=k-j}^{l-a}(k-l+1)(h+k+l-a-2i-j,j,a)' \\
\oplus & \bigoplus_{a=0}^{2l-k}\bigoplus_{i=k-l-1}^{l-a}(k-l)(h-2i-a+k-1,l+1,a)' \\
\oplus & \bigoplus_{a=0}^{2l-k}\bigoplus_{j=k-l+a-1}^{l}(k-l)\big[(h+(k-j-l+a-2),j,a)' \\
& \qquad\qquad \oplus\ (h-(k-j-l+a-2),j,a)'\big] \\
\oplus & \bigoplus_{a=0}^{2l-k}\bigoplus_{M=1}^{k-l-1}M\left(\bigoplus_{i=M-1}^{l-a}(h-a+l+M-1-2i,k-M+1,a)'\right. \\
\oplus & \bigoplus_{j=l+1}^{k-M}\Big[(h+(k+l-2M+2-a-j),j,a)' \\
& \qquad\qquad \oplus\ (h-(k+l-2M+2-a-j),j,a)'\Big] \\
\oplus & \bigoplus_{j=a+M}^{l}\Big[(h+(-k+l+a-2+2M-j),j,a)' \\
& \qquad\qquad \oplus\ (h-(-k+l+a-2+2M-j),j,a)'\Big] \\
\oplus & \left.\bigoplus_{i=l-a}^{k-a-M+1}(h+k+l-2a+1-M-2i,a-1+M,a)'\right) \\
\oplus & \bigoplus_{a=2l-k+1}^{l}\bigoplus_{j=l}^{k-l+a}\bigoplus_{i=l-a}^{k-j}(l-a+1)(h+k+l-a-j-2i,j,a)' \\
\oplus & \bigoplus_{a=2l-k+1}^{l-1}\bigoplus_{j=l}^{k-l+a+1}(l-a)\Big[(h+(k-l+a+2-j),j,a)' \\
& \qquad\qquad \oplus\ (h-(k-l+a+2-j),j,a)'\Big]
\end{aligned}
$$

$$\oplus \bigoplus_{a=2l-k+1}^{l-1} \bigoplus_{i=l-a}^{k-l+1} (l-a)(h+k-a+1-2i, l-1, a)'$$

$$\oplus \bigoplus_{a=2l-k+1}^{l-2} \bigoplus_{M=1}^{l-a-1} M \Bigg(\bigoplus_{i=l-a}^{k-a+1-M} (h+k+l-2a+1-M-2i, a-1+M, a)'$$

$$\oplus \bigoplus_{j=a+M}^{l} \Big[(h+(k-l-a+2-2M+j), j, a)' \oplus (h-(k-l-a+2-2M+j), j, a)' \Big]$$

$$\oplus \bigoplus_{j=l+1}^{k-M} \Big[(h+(k+l-a+2-2M-j), j, a)' \oplus (h-(k+l-a+2-2M-j), j, a)' \Big]$$

$$\oplus \bigoplus_{i=M-1}^{l-a} (h+l-a-1+M-2i, k-1+M, a)' \Bigg).$$

It is not difficult to see that we obtain the same geometrical pattern as discussed in section 9.3.

9.7 Examples

To end this very technical chapter, we present three explicit decompositions of kernel spaces.

The decomposition of $\mathrm{Ker}_h \mathcal{Q}_{2,2}$

$$(h,2,2)'$$

$$(h,1,1)'$$

$$(h+1,2,1)' \quad (h-1,2,1)'$$

$$
\begin{array}{ccccc}
 & & (h,0)' & & \\
 & (h+1,1,0)' & & (h-1,1,0)' & \\
(h+2,2,0)' & & (h,2,0)' & & (h-2,2,0)'
\end{array}
$$

The decomposition of $\mathrm{Ker}_h\mathcal{Q}_{3,2}$

$$
\begin{array}{ccc}
(h+1,2,2)' & & (h-1,2,2)' \\
 & (h,3,2)' &
\end{array}
$$

$$
\begin{array}{ccccc}
 & (h+1,1,1)' & & (h-1,1,1)' & \\
(h+2,2,1)' & & \underline{(h,2,1)'} & & (h-2,2,1)' \\
 & (h+1,3,1)' & & (h-1,3,1)' &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & (h+1)' & & (h-1)' & & \\
 & (h+2,1)' & & \underline{(h,1)'} & & (h-1,1)' & \\
(h+3,2)' & & \underline{(h+1,2)'} & & \underline{(h-1,2)'} & & (h-3,2)' \\
 & (h+2,3)' & & (h,3)' & & (h-2,3)' &
\end{array}
$$

The modules that are underlined once, occur twice in the decomposition.

The decomposition of $\mathrm{Ker}_h\mathcal{Q}_{4,2}$

$$
\begin{array}{ccccc}
(h+2,2,2)' & & (h,2,2)' & & (h-2,2,2)' \\
 & (h+1,3,2)' & & (h-1,3,2)' & \\
 & & (h,4,2)' & &
\end{array}
$$

$$
\begin{array}{ccccccc}
 & (h+2,1,1)' & & (h,1,1)' & & (h-2,1,1)' & \\
(h+3,2,1)' & & \underline{(h+1,2,1)'} & & \underline{(h-1,2,1)'} & & (h-3,2,1)' \\
 & (h+2,3,1)' & & \underline{(h,3,1)'} & & (h-2,3,1)' & \\
 & & (h+1,4,1)' & & (h-1,4,1)' & &
\end{array}
$$

$$\begin{array}{ccccccccc}
 & & (h+2)' & & (h)' & & (h-2)' & & \\
 & (h+3,1)' & & \underline{(h+1,1)'} & & \underline{(h-1,1)'} & & (h-3,1)' & \\
(h+4,2)' & & \underline{(h+2,2)'} & & \underline{\underline{(h,2)'}} & & \underline{(h-2,2)'} & & (h-4,2)' \\
 & (h+3,3)' & & \underline{(h+1,3)'} & & \underline{(h-1,3)'} & & (h-3,3)' & \\
 & & (h+2,4)' & & (h,4)' & & (h-2,4)' & &
\end{array}$$

The modules that are underlined once, occur twice in the decomposition; the irreducible $\mathrm{Spin}(m)$-module with highest weight $(h,2)'$ has multiplicity 3.

Chapter 10

Embedding factors

In chapter 8, a combination of an algebraic and a dimensional analysis was used to prove *which* $\mathrm{Spin}(m)$-irreducible vector spaces, denoted by their highest weight only, occur in the decomposition of the kernel space of $\mathcal{Q}_{k,l}$. The aim of this chapter is to embed these spaces into $\mathrm{Ker}_h \mathcal{Q}_{k,l}$, which is a problem similar to the one in section 4.3.2 (where $l = 0$), but more complicated.

In the first section we define the inversion operator I_Q with respect to $\mathcal{Q}_{k,l}$, which plays a fundamental role in the construction of the embedding factors. This inversion operator is the generalisation of the operator I_R of section 4.3.2. In section 10.2, new operators are introduced and a lot properties are given. By means of these results, we are able to formulate a *conjecture* of the embedding factors in section 10.3. Proving this conjecture of the embedding factors of null solutions for the higher spin Dirac operator $\mathcal{Q}_{k,l}$ (with $l > 0$) is not straightforward and the method we have been following is not the best approach to solve this problem. Ideally, the embedding factors, which can be seen as the inverted action of the dual twistor operators, are part of an algebraic structure. At the time of writing, it is not clear what the structure is, if it exists at all. However, the conjecture about the form of the embedding factors has been verified by explicit examples. The formulation of this conjecture was inspired by a result of Peter Van Lancker (see [37, 71]).

In section 10.4, the embedding factors for the Rarita-Schwinger operators are obtained using an method alternative to the one explained in chapter 4 (see also [18]). Furthermore, we have proved the conjecture for certain type A solutions in section 10.5. Finally, we have showed in section 10.6 that the conjecture holds in the case of $k = 2$ and $l = 1$.

10.1 Definitions and properties

As mentioned previously, finding embedding maps for the Spin(m)-irreducibles in $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$ is equivalent to finding the inverted action for the dual twistor operators $\langle\widetilde{\partial_u,\partial_x}\rangle$ and $\langle\partial_v,\partial_x\rangle$, which are visualised in the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}_{h-1}\,\mathcal{Q}_{k,l-1} \arrow[r, dotted, shift right] & \mathrm{Ker}_h\,\mathcal{Q}_{k,l} \arrow[l, "\langle\partial_v,\partial_x\rangle"'] \arrow[d, "\langle\widetilde{\partial_u,\partial_x}\rangle"] \\
\mathrm{Ker}_{h-2}\,\mathcal{Q}_{k-1,l-1} \arrow[ur, dotted] & \mathrm{Ker}_{h-1}\,\mathcal{Q}_{k-1,l} \arrow[u, dotted, bend left]
\end{tikzcd}$$

This means that we should be looking for an operator that goes up one degree in homogeneity in both x and u, and both x and v, respectively. Obvious choices like $\langle u,x\rangle$ and

$$\langle\widetilde{v,x}\rangle := \langle v,x\rangle(\mathbb{E}_u-\mathbb{E}_v)-\langle u,x\rangle\langle v,\partial_u\rangle, \tag{10.1}$$

which satisfies $[\langle u,\partial_v\rangle,\langle\widetilde{v,x}\rangle]=0$ when acting on the simplicial monogenics, are not suited as they are not an endomorphism of functions with values in the simplicial monogenics. We illustrate this by means of an example: if f is $\mathcal{S}_{k-1,l}$-valued, then

$$\begin{aligned}
\partial_u\langle u,x\rangle f &= xf\\
\partial_v\langle u,x\rangle f &= 0\\
\langle u,\partial_v\rangle\langle u,x\rangle f &= 0,
\end{aligned}$$

implying that

$$\langle u,x\rangle : \mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_{k-1,l})\to\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{H}_{k,l}\otimes\mathbb{S}).$$

Now, using (6.10), we construct the operator $\pi_1\langle u,x\rangle$, which is an endomorphism of functions with values in the simplicial monogenics:

$$\pi_1\langle u,x\rangle : \mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_{k-1,l})\to\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_{k,l}).$$

However, we have in general that:

$$\begin{array}{rcl}
\pi_1\langle u,x\rangle : \mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_{k-1,l}) & \to & \mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_{k,l})\\
\cup & & \cup\\
\pi_1\langle u,x\rangle : \mathrm{Ker}_{h-1}\,\mathcal{Q}_{k-1,l} & \not\to & \mathrm{Ker}_h\,\mathcal{Q}_{k,l}.
\end{array}$$

To construct an embedding map that maps solutions to solutions, we make use of the *inversion operator* with respect to $\mathcal{Q}_{k,l}$, which is defined in the next section.

10.1.1 Inversion operator

The inversion operator for spinor-valued functions, mentioned in [30, 18] and recalled in chapter 4, can easily be generalised as follows.

Definition 10. *Let $f(x;u,v) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S})$. The inversion operator I_Q corresponding to the operator $\mathcal{Q}_{k,l}$ is defined as*

$$I_Q f(x;u,v) = \frac{x}{|x|^m} f\left(\frac{x}{|x|^2}; \frac{xux}{|x|^2}, \frac{xvx}{|x|^2}\right).$$

Remark 38. *In case $l = 0$ and $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_k \otimes \mathbb{S})$, we have $I_Q f = I_R f$.*

It is obvious that $(I_Q)^2 = -\mathbf{1}$ and $\mathbb{E}_x(I_Q f) = (1-m-h)(I_Q f)$ for every f satisfying $\mathbb{E}_x f = hf$. Furthermore,

Lemma 32. *We have*

(i) *I_Q is Spin(m)-invariant*

(ii) $[I_Q, \pi_1] = 0$.

Proof.

(i) Since $|\bar{s}xs| = |x|$ for $s \in \mathrm{Spin}(m)$, we have

$$\begin{aligned}
L(s)(I_Q f) &= L(s)\frac{x}{|x|^m} f\left(\frac{x}{|x|^2}; \frac{xux}{|x|^2}, \frac{xvx}{|x|^2}\right)\\
&= s\frac{\bar{s}xs}{|x|^m} f\left(\frac{\bar{s}xs}{|x|^2}; \frac{\bar{s}xs\bar{s}us\bar{s}xs}{|x|^2}, \frac{\bar{s}xs\bar{s}vs\bar{s}xs}{|x|^2}\right)\\
&= \frac{x}{|x|^m} sf\left(\frac{\bar{s}xs}{|x|^2}; \frac{\bar{s}xuxs}{|x|^2}, \frac{\bar{s}xvxs}{|x|^2}\right)\\
&= I_Q L(s) f.
\end{aligned}$$

(ii) It is not difficult to show that $\{I_Q, u\} = 0 = \{I_Q, \partial_u\}$. Hence $[I_Q, u\partial_u] = 0$. It follows that $[I_Q, \pi_1] = 0$. □

Because the higher spin operator $\mathcal{Q}_{k,l}$ is conformally invariant, it follows from Theorem 18 in chapter 3 that this inversion operator preserves solutions.

Lemma 33. *One has*

$$I_Q : \mathrm{Ker}_h\mathcal{Q}_{k,l} \to \mathrm{Ker}_{1-m-h}\mathcal{Q}_{k,l}.$$

Proof. It follows from section 3.3.1 and (7.3) that the action of I_Q is precisely the action of the conformal group if $a=0$, $b=-1$, $c=1$ and $d=0$. Indeed, in this case, we have $g^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in V(m)$ and

$$\begin{aligned}(g\cdot f)(x) &= |x|^{-m+1}L\left(\frac{\widetilde{x}}{|x|}\right)f(-x^{-1};u,v)\\ &= \frac{x}{|x|^m}f(\frac{x}{|x|^2};\frac{xux}{|x|^2},\frac{xvx}{|x|^2}) = I_Qf(x;u,v).\end{aligned}$$

□

10.1.2 The operator $I_Q\partial_x I_Q$

In the context of finding embedding maps for null solutions of $\mathcal{Q}_{k,l}$, the operator $I_Q\partial_x I_Q$, acting on functions in $\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{H}_{k,l}\otimes\mathbb{S})$, is very important. An explicit expression is proved in the following proposition.

Proposition 26. *The operator $I_Q\partial_x I_Q$ is an endomorphism on the vector space $\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{H}_{k,l}\otimes\mathbb{S})$ and satisfies*

$$I_Q\partial_x I_Q = |x|^2\partial_x + 2\langle x,u\rangle\partial_u + 2\langle x,v\rangle\partial_v - 2u\langle x,\partial_u\rangle - 2v\langle x,\partial_v\rangle.$$

Proof. We prove this statement by explicitly calculating the action of $I_Q\partial_x I_Q$ on polynomials in $\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{H}_{k,l}\otimes\mathbb{S})$. Therefore, let $y\neq 0$ be a Clifford number and define f as a $\mathcal{H}_{k,l}$-valued polynomial of degree h in x:

$$f := \langle x,y\rangle^h \underbrace{\langle u,\mathfrak{f}_1\rangle^{k-l}\langle u\wedge v,\mathfrak{f}_1\wedge\mathfrak{f}_2\rangle^l}_{=:\,P} \tag{10.2}$$

with $\mathfrak{f}_1^2 = \mathfrak{f}_2^2 = 0$ and $\{\mathfrak{f}_1,\mathfrak{f}_2\} = 0$. For more details about the expression of elements in $\mathcal{H}_{k,l}$, we refer to [73]. An idempotent I should be added at the end of (10.2) in order to transform f into a $\mathcal{H}_{k,l}\otimes\mathbb{S}$-valued polynomial. It suffices to use the particular form (10.2) to prove the statement, because f generates the irreducible Spin(m)-module $\mathcal{H}_{k,l}$ and the operator $I_Q\partial_x I_Q$ is Spin(m)-invariant.

The next step is to calculate $I_Q f$ using Definition 10. Denote by $\mathcal{I}$ the action

$$\mathcal{I} : g(x,u,v) \to g\left(\frac{x}{|x|^2}, \frac{xux}{|x|^2}, \frac{xvx}{|x|^2}\right),$$

with $g \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S})$. This means that $I_Q = x|x|^{-m}\mathcal{I}$, when acting on polynomials in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S})$. In particular, we have $\mathcal{I}^2 = \mathbf{1}$ and

$$\begin{aligned}\mathcal{I}|x|^{-m-2(k+l)} &= |x|^{m+2(k+l)}\\ \mathcal{I}\langle x,y\rangle^h &= |x|^{-2h}\langle x,y\rangle^h\end{aligned}$$

which leads to

$$I_Q f = x|x|^{-m-2h}\langle x,y\rangle^h \underbrace{|x|^{-2(k+l)}\langle xux, \mathfrak{f}_1\rangle^{k-l}\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^l}_{= \mathcal{I}P =: \widetilde{P}}.$$

By acting with the Dirac operator ∂_x, we obtain

$$\begin{aligned}\partial_x I_Q f &= \{\partial_x, x\}|x|^{-m-2h}\langle x,y\rangle^h \widetilde{P} - x[\partial_x, |x|^{-m-2h}]\langle x,y\rangle^h \widetilde{P}\\ &\quad - x|x|^{-m-2h}[\partial_x, \langle x,y\rangle^h]\widetilde{P} - x|x|^{-m-2h}\langle x,y\rangle^h \partial_x \widetilde{P}\\ &= -hxy|x|^{-m-2h}\langle x,y\rangle^{h-1}\widetilde{P} - x|x|^{-m-2h}\langle x,y\rangle^h \partial_x \widetilde{P}.\end{aligned}$$

As we have $\mathcal{I}\widetilde{P} = \mathcal{I}^2 P = P$, acting once more with I_Q leads to the following expression:

$$\begin{aligned}I_Q \partial_x I_Q f &= x|x|^{-m}\Big(- hxy|x|^m \langle x,y\rangle^{h-1} P - x|x|^{m-2}\langle x,y\rangle^h \mathcal{I}(\partial_x \widetilde{P})\Big)\\ &= h|x|^2 y\langle x,y\rangle^{h-1} P + \langle x,y\rangle^h \mathcal{I}(\partial_x \widetilde{P})\\ &= |x|^2 \partial_x f + \langle x,y\rangle^h \mathcal{I}(\partial_x \widetilde{P}). \qquad (10.3)\end{aligned}$$

We deal with the polynomial $\mathcal{I}(\partial_x \widetilde{P})$ separately. To that end, we introduce two short notations:

$$\begin{aligned}P_1 &:= \langle u, \mathfrak{f}_1\rangle\\ P_2 &:= \langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle.\end{aligned}$$

The polynomials that are obtained by acting with $\mathcal{I}$ on P_1 and P_2, are denoted by $\widetilde{P_1}$ and $\widetilde{P_2}$, respectively:

$$\begin{aligned}\widetilde{P_1} &:= |x|^{-2}\langle xux, \mathfrak{f}_1\rangle = \langle u, \mathfrak{f}_1\rangle - 2|x|^{-2}\langle u, x\rangle\langle x, \mathfrak{f}_1\rangle \\ \widetilde{P_2} &:= |x|^{-4}\langle xux \wedge xvx, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle \\ &= \langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle - 2|x|^{-2}\langle x, u\rangle\langle x \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle \\ &\quad + 2|x|^{-2}\langle x, v\rangle\langle x \wedge u, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle.\end{aligned}$$

With these notations, we have

$$\mathcal{I}(\partial_x \widetilde{P}) = (k-l)\,\underbrace{\mathcal{I}(\partial_x \widetilde{P_1})}\,P_1^{k-l-1}{P_2}^l + l\,\underbrace{\mathcal{I}(\partial_x \widetilde{P_2})}\,{P_1}^{k-l}{P_2}^{l-1}.$$

Next, we deal with the underlined polynomials in the expression above. Straightforward calculations lead to

$$\begin{aligned}\mathcal{I}(\partial_x \widetilde{P_1}) &= -\,2u\langle x, \mathfrak{f}_1\rangle + 2\mathfrak{f}_1\langle x, u\rangle \\ \mathcal{I}(\partial_x \widetilde{P_2}) &= -\,2u\langle x \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle + 2v\langle x \wedge u, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle \\ &\quad + 2\langle x, u\rangle\big(\mathfrak{f}_1\langle v, \mathfrak{f}_2\rangle - \mathfrak{f}_2\langle v, \mathfrak{f}_1\rangle\big) \\ &\quad - 2\langle x, v\rangle\big(\mathfrak{f}_1\langle u, \mathfrak{f}_2\rangle - \mathfrak{f}_2\langle u, \mathfrak{f}_1\rangle\big).\end{aligned}$$

After substituting these expressions in $\mathcal{I}(\partial_x \widetilde{P})$, we find

$$\begin{aligned}\mathcal{I}(\partial_x \widetilde{P}) =\; & -\,2u\bigg((k-l)\langle x, \mathfrak{f}_1\rangle\langle u, \mathfrak{f}_1\rangle^{k-l-1}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^l \\ & \qquad + l\langle x \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle\langle u, \mathfrak{f}_1\rangle^{k-l}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l-1}\bigg) \\ & - 2v\bigg(\langle u \wedge x, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle\langle u, \mathfrak{f}_1\rangle^{k-l}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l-1}\bigg) \\ & + 2\langle x, u\rangle\bigg((k-l)\mathfrak{f}_1\langle u, \mathfrak{f}_1\rangle^{k-l-1}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^l \\ & + l\big(\mathfrak{f}_1\langle v, \mathfrak{f}_2\rangle - \mathfrak{f}_2\langle v, \mathfrak{f}_1\rangle\big)\bigg)\langle u, \mathfrak{f}_1\rangle^{k-l}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l-1} \\ & + 2\langle x, v\rangle l\big(\mathfrak{f}_2\langle u, \mathfrak{f}_1\rangle - \mathfrak{f}_1\langle u, \mathfrak{f}_2\rangle\big)\langle u, \mathfrak{f}_1\rangle^{k-l}\langle u \wedge v, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l-1}\end{aligned}$$

which equals precisely

$$\mathcal{I}(\partial_x \widetilde{P}) = -2u\langle x, \partial_u\rangle P - 2v\langle x, \partial_v\rangle P + 2\langle x, u\rangle\partial_u P + 2\langle x, v\rangle\partial_v P.$$

Together with (10.3), this leads to the desired result that

$$I_Q\partial_x I_Q f = |x|^2\partial_x f + 2\langle x,u\rangle\partial_u f + 2\langle x,v\rangle\partial_v f - 2u\langle x,\partial_u\rangle f - 2v\langle x,\partial_v\rangle f.$$

It is not difficult to see that $I_Q\partial_x I_Q f$ is again $\mathcal{H}_{k,l}\otimes\mathbb{S}$-valued. □

Corollary 3. *For integers $h\geq k\geq l$, one has $I_Q\partial_x I_Q\mathcal{S}_{h,k,l}=0$.*

Proof. This follows immediately from the definition of $\mathcal{S}_{h,k,l}$. □

In view of Corollary 3 and the fact that $I_Q\partial_x I_Q$ acts on $\mathcal{H}_{k,l}\otimes\mathbb{S}$-valued functions, the anti-commutator of $I_Q\partial_x I_Q$ and the embedding factors u and $\widetilde{v}$ are important, respectively, as we have

$$\begin{aligned} I_Q\partial_x I_Q u\mathcal{S}_{k-1,l} &= \{I_Q\partial_x I_Q, u\}\mathcal{S}_{k-1,l}\\ I_Q\partial_x I_Q \widetilde{v}\mathcal{S}_{k,l-1} &= \{I_Q\partial_x I_Q, \widetilde{v}\}\mathcal{S}_{k,l-1}\end{aligned}$$

and it follows from a direct calculation that

Lemma 34. *One has*

$$\begin{aligned}\{I_Q\partial_x I_Q,u\} =& -2|x|^2\langle u,\partial_x\rangle - 2\langle x,u\rangle(m+2\mathbb{E}_u) - 2ux\\ &+ 4|u|^2\langle x,\partial_u\rangle - 4\langle x,v\rangle\langle u,\partial_v\rangle + 4\langle u,v\rangle\langle x,\partial_v\rangle\\ \{I_Q\partial_x I_Q,\widetilde{v}\} =& -2|x|^2\langle\widetilde{v,\partial_x}\rangle - 2\langle\widetilde{x,v}\rangle(m+2\mathbb{E}_v-2) - 2\widetilde{v}x\\ &+ 4|v|^2\langle x,\partial_v\rangle(\mathbb{E}_u-\mathbb{E}_v) - 4|u|^2\langle v,\partial_u\rangle\langle x,\partial_u\rangle\\ &+ 4\langle u,v\rangle\left(\langle x,\partial_u\rangle(\mathbb{E}_u-\mathbb{E}_v) - \langle x,\partial_v\rangle\langle v,\partial_u\rangle\right)\\ &+ 4\langle x,v\rangle\langle v,\partial_u\rangle\langle u,\partial_v\rangle.\end{aligned}$$

Proof. Use the (anti-)commutator identities of products of operators in the beginning of section 10.2. □

The result of Lemma 26 can be rewritten in such a way that it features embedding factors u and $\widetilde{v}$ (instead of v):

$$\begin{aligned} I_Q\partial_x I_Q =& |x|^2\partial_x + 2\langle x,u\rangle\partial_u + 2\langle x,v\rangle\partial_v\\ &- 2u\langle\widetilde{x,\partial_u}\rangle(\mathbb{E}_u-\mathbb{E}_v+1)^{-1} - 2\widetilde{v}\langle x,\partial_v\rangle(\mathbb{E}_u-\mathbb{E}_v+1)^{-1} \end{aligned} \tag{10.4}$$

where we have introduced the operator

$$\langle\widetilde{x,\partial_u}\rangle := \langle x,\partial_u\rangle(\mathbb{E}_u-\mathbb{E}_v+1) + \langle v,\partial_u\rangle\langle x,\partial_v\rangle, \tag{10.5}$$

which commutes with $\langle u, \partial_v \rangle$ when acting on polynomials with values in the simplicial monogenics.

The next result will be useful in what follows.

Lemma 35. *The operator $\{\partial_x, I_Q \partial_x I_Q\}$ is an endomorphism of functions with values in the simplicial monogenics in two variables, i.e.*

$$\{\partial_x, I_Q \partial_x I_Q\} \in \mathrm{End}(\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}))$$

for integers $k \geq l$.

Proof. If f is a function with values in $\mathcal{S}_{k,l}$, then

$$\begin{aligned}\{\partial_x, I_Q \partial_x I_Q\} f = {} & 2x\partial_x f + 2\langle x, u\rangle \partial_u \partial_x f + 2\langle x, v\rangle \partial_v \partial_x f \\ & - 2|x|^2 \Delta_x f + 4\langle u, \partial_x\rangle\langle x, \partial_u\rangle f + 4\langle v, \partial_x\rangle\langle x, \partial_v\rangle f. \end{aligned} \tag{10.6}$$

Due to the symmetry in u and v in the above expression, it suffices to verify that $\partial_u\{\partial_x, I_Q \partial_x I_Q\} f = 0$. Indeed, the action with ∂_u on the right-hand side of (10.6) leads to

$$2\{\partial_u, x\}\partial_x f - 2x\partial_u\partial_x f + 2x\partial_u\partial_x f + 4\partial_x\langle x, \partial_u\rangle f = 0.$$

Hence, we also have $\partial_v\{\partial_x, I_Q \partial_x I_Q\} f = 0$. Rewriting (10.6) as

$$\begin{aligned}\{\partial_x, I_Q \partial_x I_Q\} f = {} & 2x\partial_x f - 4\langle x, u\rangle \widetilde{\langle \partial_u, \partial_x\rangle}(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} f \\ & - 4\widetilde{\langle x, v\rangle}\langle \partial_v, \partial_x\rangle(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} f \\ & - 2|x|^2\Delta_x f + 4\langle u, \partial_x\rangle\widetilde{\langle x, \partial_u\rangle}(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} f \\ & + 4\widetilde{\langle v, \partial_x\rangle}\langle x, \partial_v\rangle(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} f \end{aligned}$$

it immediately follows that $\langle u, \partial_v\rangle\{\partial_x, I_Q \partial_x I_Q\} f = 0$. $\square$

We conclude this section by remarking that the result of Lemma 26 can be written by means of the Gamma operator:

Corollary 4. *The operator $I_Q \partial_x I_Q$ satisfies*

$$I_Q \partial_x I_Q = |x|^2 \partial_x - [\Gamma_u + \Gamma_v, x].$$

Proof. A direct calculation leads to the above statement:

$$\begin{aligned}
[\Gamma_u, x] &= -\sum_{i<j}\sum_k [e_{ij}(u_i\partial_{u_j} - u_j\partial_{u_i}), e_k x_k] \\
&= -\sum_{i<j}\sum_k [e_{ij}, e_k](u_i\partial_{u_j} - u_j\partial_{u_i})x_k \\
&= -\sum_{i<j}\sum_k (e_i\{e_j, e_k\} - \{e_i, e_k\}e_j)(u_i\partial_{u_j} - u_j\partial_{u_i})x_k \\
&= -\sum_{i<j}\sum_k (-2e_i\delta_{jk} + 2\delta_{ik}e_j)(u_i\partial_{u_j} - u_j\partial_{u_i})x_k \\
&= 2\sum_{i\neq j} e_i(u_i\partial_{u_j} - u_j\partial_{u_i})x_j \\
&= 2u\langle x, \partial_u\rangle - 2\langle x, u\rangle\partial_u.
\end{aligned}$$

□

Remark 39. *This generalises a result in [71], which states that*

$$I_R\partial_x I_R = |x|^2\partial_x - [\Gamma_u, x].$$

10.2 Useful results

In this section, we list an overview of (anti-)commutators involving operators that play an important role in this thesis, such as u, $\widetilde{v}$, $\langle \widetilde{u, v}\rangle$, $\langle \widetilde{\partial_u, \partial_x}\rangle$, ∂_v, $\langle \widetilde{x, \partial_u}\rangle$, $I_Q\partial_x I_Q$ and many others. These results will probably be useful when we are trying to discover an underlying structure to the embedding factors.

An efficient way to calculate (anti-)commutators of products of operators is by using the identities

$$\begin{aligned}
\{AB, C\} &= A[B, C] + \{A, C\}B = A\{B, C\} - [A, C]B \\
\{A, BC\} &= [A, B]C + B\{A, C\} = \{A, B\}C - B[A, C] \\
[AB, CD] &= AC[B, D] + A[B, C]D + [A, C]DB + C[A, D]B \\
&= -AC\{B, D\} + A\{B, C\}D + \{A, C\}DB - C\{A, D\}B \\
\{AB, CD\} &= A[B, C]D + \{A, C\}BD - CA[B, D] - C[A, D]B.
\end{aligned}$$

Another useful result, which holds for every $u, x \in \mathbb{R}^m$, is

$$[\langle \partial_u, \partial_x \rangle, \langle u, x \rangle] = m + \mathbb{E}_x + \mathbb{E}_u. \tag{10.7}$$

As the following expression occurs frequently in the (anti-)commutators, we denote it by $\Omega_{u,v}$:

$$\Omega_{u,v} := \left(\widetilde{v} \langle u, \partial_x \rangle - u \langle \widetilde{v, \partial_x} \rangle \right) (\mathbb{E}_u - \mathbb{E}_v + 1)^{-1}. \tag{10.8}$$

Finally, we introduce the short notation

$$\widetilde{\partial_u} \ := \ \partial_u (\mathbb{E}_u - \mathbb{E}_v) + \partial_v \langle v, \partial_u \rangle. \tag{10.9}$$

Results involving u, $\widetilde{v}$ and $\widetilde{\partial_u}$, ∂_v

It is not difficult to prove that

Lemma 36. *One has*

$$\begin{aligned}
\{\widetilde{\partial_u}, u\} &= -(m + 2\mathbb{E}_u)(\mathbb{E}_u - \mathbb{E}_v + 2) - v\partial_v - u\partial_u - 2\langle v, \partial_u \rangle \langle u, \partial_v \rangle \\
\{\partial_u, \widetilde{v}\} &= (m + 2\mathbb{E}_v - 4)\langle v, \partial_u \rangle - v\partial_u \\
\{\widetilde{\partial_u}, \widetilde{v}\} &= 0 \\
\{\partial_v, \widetilde{v}\} &= -(m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v) + v\partial_v + u\partial_u - 2\langle v, \partial_u \rangle \langle u, \partial_v \rangle.
\end{aligned}$$

Lemma 37. *The action of the Dirac operator on the product of vu and $\widetilde{v}u$, respectively, is given by*

$$\begin{aligned}
[\partial_x, vu] &= 2\Omega_{u,v} \\
[\partial_x, \widetilde{v}u] &= 2\Omega_{u,v}(\mathbb{E}_u - \mathbb{E}_v + 2) \\
[\partial_x, u\widetilde{v}] &= -2\Omega_{u,v}(\mathbb{E}_u - \mathbb{E}_v) \\
[\partial_x, \langle \widetilde{u, v} \rangle] &= 2\Omega_{u,v}(m + \mathbb{E}_u + \mathbb{E}_v - 1).
\end{aligned}$$

Proof. The last statement immediately follows from the definition of $\langle \widetilde{u, v} \rangle$ and the first line. ☐

Results involving $\langle u, \partial_x \rangle$ and $\langle \widetilde{v, \partial_x} \rangle$

The operators $\langle u, \partial_x \rangle$ and

$$\langle \widetilde{v, \partial_x} \rangle := \langle v, \partial_x \rangle (\mathbb{E}_u - \mathbb{E}_v) - \langle u, \partial_x \rangle \langle v, \partial_u \rangle,$$

which were introduced already in (2.54) and (2.55), respectively, occur in the expression of the anti-commutators $\{I_Q\partial_x I_Q, u\}$ and $\{I_Q\partial_x I_Q, \widetilde{v}\}$. It is convenient to have an expression for the commutator of these operators with other important operators, such as u, $\widetilde{v}$, ∂_v and the dual twistor operators.

Lemma 38. *The action of the embedding factors u and $\widetilde{v}$ on $\langle u, \partial_x\rangle$ and $\widetilde{\langle v, \partial_x\rangle}$ is given by*

$$[u, \langle u, \partial_x\rangle] = 0 \qquad [\widetilde{v}, \langle u, \partial_x\rangle] = \Omega_{u,v}$$
$$[u, \widetilde{\langle v, \partial_x\rangle}] = \Omega_{u,v} \qquad [\widetilde{v}, \widetilde{\langle v, \partial_x\rangle}] = 0.$$

Remark 40. *The following commutators are closely related to each other:*

$$\big[\widetilde{v}, \langle u, \partial_x\rangle\big] = \big[u, \widetilde{\langle v, \partial_x\rangle}\big] = \frac{1}{2}\big[\partial_x, \widetilde{v}u\big](\mathbb{E}_u - \mathbb{E}_v + 2)^{-1} = \frac{1}{2}\big[\partial_x, vu\big] = \Omega_{u,v}.$$

Next, we present two lemmas that consist of very elegant results.

Lemma 39. *The action of the operators $\widetilde{\partial_u}$ and ∂_v on $\langle u, \partial_x\rangle$ and $\widetilde{\langle v, \partial_x\rangle}$ is given by*

$$\begin{aligned}
&[\widetilde{\partial_u}, \langle u, \partial_x\rangle] = (\mathbb{E}_u - \mathbb{E}_v + 2)\partial_x + \langle u, \partial_x\rangle\partial_u + \langle v, \partial_x\rangle\partial_v\\
&[\widetilde{\partial_u}, \widetilde{\langle v, \partial_x\rangle}] = 0\\
&[\partial_v, \langle u, \partial_x\rangle] = 0\\
&[\partial_v, \widetilde{\langle v, \partial_x\rangle}] = (\mathbb{E}_u - \mathbb{E}_v)\partial_x - \langle u, \partial_x\rangle\partial_u - \langle v, \partial_x\rangle\partial_v.
\end{aligned}$$

Lemma 40. *The action of the dual twistors $\widetilde{\langle \partial_u, \partial_x\rangle}$ and $\langle \partial_v, \partial_x\rangle$ on $\langle u, \partial_x\rangle$ and $\widetilde{\langle v, \partial_x\rangle}$ is given by*

$$\begin{aligned}
&[\widetilde{\langle \partial_u, \partial_x\rangle}, \langle u, \partial_x\rangle] = \Delta_x(\mathbb{E}_u - \mathbb{E}_v + 2) + \langle u, \partial_x\rangle\langle \partial_u, \partial_x\rangle + \widetilde{\langle v, \partial_x\rangle}\langle \partial_v, \partial_x\rangle\\
&[\widetilde{\langle \partial_u, \partial_x\rangle}, \widetilde{\langle v, \partial_x\rangle}] = 0\\
&[\langle \partial_v, \partial_x\rangle, \langle u, \partial_x\rangle] = 0\\
&[\langle \partial_v, \partial_x\rangle, \widetilde{\langle v, \partial_x\rangle}] = \Delta_x(\mathbb{E}_u - \mathbb{E}_v) - \langle u, \partial_x\rangle\langle \partial_u, \partial_x\rangle - \widetilde{\langle v, \partial_x\rangle}\langle \partial_v, \partial_x\rangle.
\end{aligned}$$

Results involving $\widetilde{\langle \partial_u, \partial_x\rangle}$, $\langle \partial_v, \partial_x\rangle$ and $I_Q\partial_x I_Q$

Lemma 41. *Acting on $\mathcal{H}_{k,l} \otimes \mathbb{S}$-valued functions, one has*

$$\begin{aligned}
\big[\widetilde{\langle \partial_u, \partial_x\rangle}, I_Q\partial_x I_Q\big] &= 2(m + \mathbb{E}_x + \mathbb{E}_u - 1)\widetilde{\partial_u}\\
\big[\langle \partial_v, \partial_x\rangle, I_Q\partial_x I_Q\big] &= 2(m + \mathbb{E}_x + \mathbb{E}_v - 2)\partial_v.
\end{aligned}$$

Using these results, it immediately follows from Lemma 36 that

Corollary 5. *Acting on polynomials in $\mathcal{M}^s_{h,k,l}$, one has*

$$
\begin{aligned}
\widetilde{\langle \partial_u, \partial_x \rangle} I_Q \partial_x I_Q \widetilde{v} &= 0\\
\widetilde{\langle \partial_u, \partial_x \rangle} I_Q \partial_x I_Q u &= -2(m + \mathbb{E}_x + \mathbb{E}_u - 1)(m + 2\mathbb{E}_u)(\mathbb{E}_u - \mathbb{E}_v + 2)\mathcal{M}^s_{h,k,l}\\
\langle \partial_v, \partial_x \rangle I_Q \partial_x I_Q u &= 0\\
\langle \partial_v, \partial_x \rangle I_Q \partial_x I_Q \widetilde{v} &= -2(m + \mathbb{E}_x + \mathbb{E}_v - 2)(m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v)\mathcal{M}^s_{h,k,l}.
\end{aligned}
$$

Results involving $I_Q\langle u, \partial_x \rangle I_Q$ and $I_Q\widetilde{\langle v, \partial_x \rangle} I_Q$

Not only do they occur in the anti-commutators $\{I_Q \partial_x I_Q, u\}$ and $\{I_Q \partial_x I_Q, \widetilde{v}\}$, there is another obvious connection between the operators $\langle u, \partial_x \rangle$ and $\widetilde{\langle v, \partial_x \rangle}$ and the operators in Lemma 34:

Lemma 42. *One has*

$$
\begin{aligned}
I_Q\langle u, \partial_x \rangle I_Q &= \frac{1}{2}\{I_Q \partial_x I_Q, u\}\\
&= -|x|^2\langle u, \partial_x \rangle - \langle x, u \rangle(m + 2\mathbb{E}_u) - ux\\
&\quad + 2|u|^2\langle x, \partial_u \rangle - 2\langle x, v \rangle\langle u, \partial_v \rangle + 2\langle u, v \rangle\langle x, \partial_v \rangle\\
I_Q\widetilde{\langle v, \partial_x \rangle} I_Q &= \frac{1}{2}\{I_Q \partial_x I_Q, \widetilde{v}\}\\
&= -|x|^2\widetilde{\langle v, \partial_x \rangle} - \widetilde{\langle x, v \rangle}(m + 2\mathbb{E}_v - 2) - \widetilde{v}x\\
&\quad + 2|v|^2\langle x, \partial_v \rangle(\mathbb{E}_u - \mathbb{E}_v) - 2|u|^2\langle v, \partial_u \rangle\langle x, \partial_u \rangle\\
&\quad + 2\langle u, v \rangle\left(\langle x, \partial_u \rangle(\mathbb{E}_u - \mathbb{E}_v) - \langle x, \partial_v \rangle\langle v, \partial_u \rangle\right)\\
&\quad + 2\langle x, v \rangle\langle v, \partial_u \rangle\langle u, \partial_v \rangle.
\end{aligned}
$$

Proof. This follows from Lemma 34 together with $\{I_Q, u\} = 0 = \{I_Q, \widetilde{v}\}$. $\square$

The statements of the next lemma will turn out to be very useful in what follows.

Lemma 43. *The action of the operators ∂_x and $I_Q \partial_x I_Q$ on $I_Q\langle u, \partial_x \rangle I_Q$ and*

$I_Q\langle \widetilde{v,\partial_x}\rangle I_Q$ is given by

$$\begin{aligned}
[\partial_x, I_Q\langle u,\partial_x\rangle I_Q] &= -2u(m+\mathbb{E}_x+\mathbb{E}_u-1)+2|u|^2\partial_u+2\langle u,v\rangle\partial_v-2v\langle u,\partial_v\rangle\\
[\partial_x, I_Q\langle \widetilde{v,\partial_x}\rangle I_Q] &= -2\widetilde{v}(m+\mathbb{E}_x+\mathbb{E}_v-2)+2|v|^2\partial_v(\mathbb{E}_u-\mathbb{E}_v)\\
&\quad -2|u|^2\langle v,\partial_u\rangle\partial_u+2\langle u,v\rangle\partial_u(\mathbb{E}_u-\mathbb{E}_v-1)\\
&\quad +2\langle u,v\rangle\langle v,\partial_u\rangle\partial_v+2v\langle v,\partial_u\rangle\langle u,\partial_v\rangle\\
[I_Q\partial_x I_Q, \langle u,\partial_x\rangle] &= -2u(\mathbb{E}_x-\mathbb{E}_u)-2|u|^2\partial_u-2\langle u,v\rangle\partial_v+2v\langle u,\partial_v\rangle\\
[I_Q\partial_x I_Q, \langle \widetilde{v,\partial_x}\rangle] &= -2\widetilde{v}(\mathbb{E}_x-\mathbb{E}_v+1)-2|v|^2\partial_v(\mathbb{E}_u-\mathbb{E}_v)\\
&\quad +2|u|^2\langle v,\partial_u\rangle\partial_u-2\langle u,v\rangle\partial_u(\mathbb{E}_u-\mathbb{E}_v-1)\\
&\quad -2\langle u,v\rangle\langle v,\partial_u\rangle\partial_v-2v\langle v,\partial_u\rangle\langle u,\partial_v\rangle.
\end{aligned}$$

Furthermore,

Lemma 44. *The action of the operators $\widetilde{\partial_u}$ and ∂_v on the operators $I_Q\langle u,\partial_x\rangle I_Q$ and $I_Q\langle \widetilde{v,\partial_x}\rangle I_Q$ is given by*

$$\begin{aligned}
[\widetilde{\partial_u}, I_Q\langle u,\partial_x\rangle I_Q] &= -(\mathbb{E}_u-\mathbb{E}_v+2)I_Q\partial_x I_Q+I_Q\langle v,\partial_x\rangle I_Q\partial_v\\
&\quad +I_Q\langle u,\partial_x\rangle I_Q\partial_u\\
[\widetilde{\partial_u}, I_Q\langle \widetilde{v,\partial_x}\rangle I_Q] &= 0\\
[\partial_v, I_Q\langle u,\partial_x\rangle I_Q] &= 0\\
[\partial_v, I_Q\langle \widetilde{v,\partial_x}\rangle I_Q] &= -(\mathbb{E}_u-\mathbb{E}_v)I_Q\partial_x I_Q-I_Q\langle v,\partial_x\rangle I_Q\partial_v\\
&\quad -I_Q\langle u,\partial_x\rangle I_Q\partial_u\\
[\partial_u, I_Q\langle u,\partial_x\rangle I_Q] &= -I_Q\partial_x I_Q.
\end{aligned}$$

Proof. This follows from $\{I_Q,\partial_u\}=0$ and Lemma 39. □

If we let the commutators in Lemma 44 act on polynomials in the vector space $\mathcal{S}_{h,k,l}$, which determines a special solution of $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$, the operators $\widetilde{\partial_u}$ and ∂_v commute with the operators $I_Q\langle u,\partial_x\rangle I_Q$ and $I_Q\langle \widetilde{v,\partial_x}\rangle I_Q$. We can state this as follows:

Corollary 6. *For all integers $h\geq k\geq l$, one has*

$$\begin{aligned}
\pi_1 I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h,k,l} &= I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h,k,l}\\
\pi_1 I_Q\langle \widetilde{v,\partial_x}\rangle I_Q\mathcal{S}_{h,k,l} &= I_Q\langle \widetilde{v,\partial_x}\rangle I_Q\mathcal{S}_{h,k,l}.
\end{aligned}$$

Proof. The second result follows immediately from Lemma 44 and Corollary 3:

$$\begin{aligned}
\partial_u I_Q \widetilde{\langle v, \partial_x \rangle} I_Q \mathcal{S}_{h,k,l} &= -I_Q \partial_x I_Q \mathcal{S}_{h,k,l} = 0 \\
\partial_v I_Q \widetilde{\langle v, \partial_x \rangle} I_Q \mathcal{S}_{h,k,l} &= [\partial_v, I_Q \widetilde{\langle v, \partial_x \rangle} I_Q] \mathcal{S}_{h,k,l} = 0 \\
\langle u, \partial_v \rangle I_Q \widetilde{\langle v, \partial_x \rangle} I_Q \mathcal{S}_{h,k,l} &= 0.
\end{aligned}$$

The first statement is proved in a similar way. □

More elegant results are obtained if we consider the action of the dual twistors on the operators of interest.

Lemma 45. *Acting on polynomials with values in the simplicial monogenics, i.e. polynomials in the kernel of $\partial_u, \partial_v, \langle u, \partial_v \rangle$, one has*

$$\begin{aligned}
\left[\widetilde{\langle \partial_u, \partial_x \rangle}, I_Q \langle u, \partial_x \rangle I_Q\right] &= -(m + \mathbb{E}_x + \mathbb{E}_u - 1)(m + 2\mathbb{E}_u)(\mathbb{E}_u - \mathbb{E}_v + 2) \\
\left[\widetilde{\langle \partial_u, \partial_x \rangle}, I_Q \widetilde{\langle v, \partial_x \rangle} I_Q\right] &= 0 \\
\left[\langle \partial_v, \partial_x \rangle, I_Q \widetilde{\langle v, \partial_x \rangle} I_Q\right] &= -(m + \mathbb{E}_x + \mathbb{E}_v - 2)(m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v) \\
\left[\langle \partial_v, \partial_x \rangle, I_Q \langle u, \partial_x \rangle I_Q\right] &= 0.
\end{aligned}$$

Proof. This follows from Lemma 43 and Lemma 44. □

Finally, we present another lemma that consists of useful results, in particular for the calculations in section 10.6.

Lemma 46. *One has*

$$\begin{aligned}
[\langle u, \partial_x \rangle, I_Q \langle u, \partial_x \rangle I_Q] &= -|u|^2 (m + 2\mathbb{E}_x - 2) \\
[\widetilde{\langle v, \partial_x \rangle}, I_Q \langle u, \partial_x \rangle I_Q] &= \{u, \widetilde{v}\}(m + \mathbb{E}_x + \mathbb{E}_u - 1) + [|u|^2, \widetilde{v}]\partial_u - \langle u, v \rangle \{\partial_v, \widetilde{v}\} \\
&\quad + \{v, \widetilde{v}\}\partial_v + [\langle u, v \rangle, \widetilde{v}]\partial_v - |u|^2 \{\partial_u, \widetilde{v}\}.
\end{aligned}$$

Proof. These results are obtained using Lemma 43. Acting on functions with values in the simplicial monogenics, the second commutator reduces to

$$\begin{aligned}
[\widetilde{\langle v, \partial_x \rangle}, I_Q \langle u, \partial_x \rangle I_Q] &= \{u, \widetilde{v}\}(m + \mathbb{E}_x + \mathbb{E}_u - 1) - \langle u, v \rangle \{\partial_v, \widetilde{v}\} - |u|^2 \{\partial_u, \widetilde{v}\} \\
&= \{u, \widetilde{v}\}(m + \mathbb{E}_x + \mathbb{E}_u - 1) \\
&\quad + \langle u, v \rangle (m + 2\mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v) \\
&\quad - |u|^2 \langle v, \partial_u \rangle (m + 2\mathbb{E}_v - 2).
\end{aligned}$$

□

Results involving the embedding factor $\langle \widetilde{u,v} \rangle$

Because the factor $\langle \widetilde{u,v} \rangle$, which occurs in Proposition 27 (see later), is responsible for making calculations very involved, it is an efficient strategy to calculate the commutator with the other common operators.

Lemma 47. *The action of ∂_u, ∂_v and $\langle u, \partial_v \rangle$ on $\langle \widetilde{u,v} \rangle$ is given by*

$$\begin{aligned}
[\partial_u, \langle \widetilde{u,v} \rangle] &= 2\langle u,v\rangle\partial_u + v(m+2\mathbb{E}_u-2)(m+\mathbb{E}_u+\mathbb{E}_v-1)\\
&\quad - 2u\langle v,\partial_u\rangle(m+\mathbb{E}_u+\mathbb{E}_v-1) + vu\partial_u\\
\big[\partial_v, \langle \widetilde{u,v} \rangle\big] &= -u(m+2\mathbb{E}_v-2)(m+\mathbb{E}_u+\mathbb{E}_v-1) + vu\partial_v - |u|^2\partial_u\\
\big[\langle v,\partial_u\rangle, \langle \widetilde{u,v} \rangle\big] &= |v|^2(\mathbb{E}_u - \mathbb{E}_v).
\end{aligned}$$

In particular,

Corollary 7. *Acting on functions with values in the simplicial monogenics, the commutator of $\widetilde{\partial_u}$ and ∂_v with $\langle \widetilde{u,v} \rangle$ is given by*

$$\begin{aligned}
\big[\partial_v, \langle \widetilde{u,v} \rangle\big] &= -u(m+2\mathbb{E}_v-2)(m+\mathbb{E}_u+\mathbb{E}_v-1)\\
\big[\widetilde{\partial_u}, \langle \widetilde{u,v} \rangle\big] &= \widetilde{v}(m+2\mathbb{E}_u)(m+\mathbb{E}_u+\mathbb{E}_v-1).
\end{aligned}$$

Lemma 48. *The action of the embedding factors u and $\widetilde{v}$ on $\langle \widetilde{u,v} \rangle$ is given by*

$$\begin{aligned}
\big[\langle \widetilde{u,v} \rangle, u\big] &= 2u(u\wedge v)(m+\mathbb{E}_u+\mathbb{E}_v-1)\\
\big[\langle \widetilde{u,v} \rangle, \widetilde{v}\big] &= -2\widetilde{v}(u\wedge v)(m+\mathbb{E}_u+\mathbb{E}_v-1).
\end{aligned}$$

The operator $\Omega_{u,v}$ occurs once again:

Lemma 49. *The commutator of the operators $\langle u, \partial_x \rangle$, $\langle v, \partial_x \rangle$ and $\langle \widetilde{v, \partial_x} \rangle$ with $\langle \widetilde{u,v} \rangle$ is given by*

$$\begin{aligned}
\big[\langle u,\partial_x\rangle, \langle \widetilde{u,v} \rangle\big] &= u\Omega_{u,v}\\
\big[\langle v,\partial_x\rangle, \langle \widetilde{u,v} \rangle\big] &= -vu\langle v,\partial_x\rangle\\
\big[\langle \widetilde{v,\partial_x}\rangle, \langle \widetilde{u,v} \rangle\big] &= \widetilde{v}\Omega_{u,v}
\end{aligned}$$

Unfortunately, the action with the twistor operators leads to an expression that is not so elegant.

Lemma 50. *The action of the twistor operators on $\langle \widetilde{u,v} \rangle$ is given by*

$$
\begin{aligned}
[\langle \widetilde{\partial_u, \partial_x} \rangle, \langle \widetilde{u,v} \rangle] &= \langle \widetilde{v, \partial_x} \rangle (m + 2\mathbb{E}_u) + \widetilde{v}\partial_x (m + \mathbb{E}_u + \mathbb{E}_v - 2) + vu \langle \widetilde{\partial_u, \partial_x} \rangle \\
&\quad + 2\langle u, v \rangle \langle \partial_u, \partial_x \rangle (\mathbb{E}_u - \mathbb{E}_v) + v^2 \langle \partial_v, \partial_x \rangle \\
\big[\langle \partial_v, \partial_x \rangle, \langle \widetilde{u,v} \rangle\big] &= -\langle u, \partial_x \rangle (m + 2\mathbb{E}_v - 2) - u\partial_x (m + \mathbb{E}_u + \mathbb{E}_v - 2) \\
&\quad + vu \langle \partial_v, \partial_x \rangle + u^2 \langle \partial_u, \partial_x \rangle .
\end{aligned}
$$

Results involving $\langle \widetilde{x, \partial_u} \rangle$ and $\langle x, \partial_v \rangle$

Recall from (10.5) the definition of the operator $\langle \widetilde{x, \partial_u} \rangle$. It is not difficult to verify that

$$\big[\langle \widetilde{x, \partial_u} \rangle, \langle x, \partial_v \rangle\big] = 0.$$

Furthermore,

Lemma 51. *One has*

$$
\begin{aligned}
\big[\langle \widetilde{x, \partial_u} \rangle, \langle u, \partial_x \rangle\big] &= (\mathbb{E}_x - \mathbb{E}_u)(\mathbb{E}_u - \mathbb{E}_v + 2) + \langle u, \partial_x \rangle \langle x, \partial_u \rangle \\
&\quad + \langle v, \partial_x \rangle \langle x, \partial_v \rangle - \langle v, \partial_u \rangle \langle u, \partial_v \rangle \\
\big[\langle \widetilde{x, \partial_u} \rangle, \langle \widetilde{v, \partial_x} \rangle\big] &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\big[\langle x, \partial_v \rangle, \langle u, \partial_x \rangle\big] &= -\langle u, \partial_v \rangle \\
\big[\langle x, \partial_v \rangle, \langle \widetilde{v, \partial_x} \rangle\big] &= (\mathbb{E}_x - \mathbb{E}_v + 1)(\mathbb{E}_u - \mathbb{E}_v) - \langle u, \partial_x \rangle \langle x, \partial_u \rangle \\
&\quad - \langle v, \partial_x \rangle \langle x, \partial_v \rangle + \langle v, \partial_u \rangle \langle u, \partial_v \rangle .
\end{aligned}
$$

In particular,

Lemma 52. *Acting on functions with values in the simplicial monogenics in two variables, one has*

$$
\begin{aligned}
\big[\langle \widetilde{x, \partial_u} \rangle, \langle u, \partial_x \rangle^n\big] &= n\langle u, \partial_x \rangle^{n-1} \left(\mathbb{E}_x - \mathbb{E}_u - \frac{n-1}{2} \right) (\mathbb{E}_u - \mathbb{E}_v + n + 1) \\
\big[\langle \widetilde{x, \partial_u} \rangle, \langle \widetilde{v, \partial_x} \rangle^n\big] &= 0 \\
\big[\langle x, \partial_v \rangle, \langle u, \partial_x \rangle^n\big] &= 0 \\
\big[\langle x, \partial_v \rangle, \langle \widetilde{v, \partial_x} \rangle^n\big] &= n\langle \widetilde{v, \partial_x} \rangle^{n-1} \left(\mathbb{E}_x - \mathbb{E}_v - \frac{n-3}{2} \right) (\mathbb{E}_u - \mathbb{E}_v - n + 1).
\end{aligned}
$$

for every positive integer n.

Proof. The proof of the first and last statement goes by induction on n. The case $n = 1$ is presented in Lemma 51. $\square$

Next, several commutators with elegant outcomes are featured in the following two lemmas.

Lemma 53. *The commutator of $\widetilde{\langle x, \partial_u \rangle}$ and $\langle x, \partial_v \rangle$ with $I_Q \partial_x I_Q$ is given by*

$$\begin{aligned}
\left[\widetilde{\langle x, \partial_u \rangle}, I_Q \partial_x I_Q\right] &= |x|^2 \widetilde{\partial_u} - 2x \widetilde{\langle x, \partial_u \rangle} \\
\left[\langle x, \partial_v \rangle, I_Q \partial_x I_Q\right] &= |x|^2 \partial_v - 2x \langle x, \partial_v \rangle.
\end{aligned}$$

Lemma 54. *The action of $\widetilde{\langle x, \partial_u \rangle}$ and $\langle x, \partial_v \rangle$ on the embedding factors u and $\widetilde{v}$ is given by*

$$\begin{aligned}
\left[\widetilde{\langle x, \partial_u \rangle}, u\right] &= x(\mathbb{E}_u - \mathbb{E}_v - 2) + u\langle x, \partial_u \rangle + v\langle x, \partial_v \rangle \\
\left[\widetilde{\langle x, \partial_u \rangle}, \widetilde{v}\right] &= 0 \\
\left[\langle x, \partial_v \rangle, u\right] &= 0 \\
\left[\langle x, \partial_v \rangle, \widetilde{v}\right] &= x(\mathbb{E}_u - \mathbb{E}_v) - u\langle x, \partial_u \rangle - v\langle x, \partial_v \rangle.
\end{aligned}$$

Finally, also the following commutators will be useful to calculate results involving embedding factors.

Lemma 55. *Acting on the kernel of ∂_u, ∂_v and $\langle u, \partial_v \rangle$, one has*

$$\begin{aligned}
[\widetilde{\langle x, \partial_u \rangle}, I_Q \langle u, \partial_x \rangle I_Q] &= 2\langle u, x \rangle \widetilde{\langle x, \partial_u \rangle} - |x|^2 (m + \mathbb{E}_x + \mathbb{E}_u - 1)(\mathbb{E}_u - \mathbb{E}_v + 2) \\
&\quad + 2\langle u, x \rangle \langle x, \partial_u \rangle + 2\langle v, x \rangle \langle x, \partial_v \rangle \\
&\quad + I_Q \langle v, \partial_x \rangle I_Q \langle x, \partial_v \rangle + I_Q \langle u, \partial_x \rangle I_Q \langle x, \partial_u \rangle \\
[\langle x, \partial_v \rangle, I_Q \widetilde{\langle v, \partial_x \rangle} I_Q] &= 2\widetilde{\langle v, x \rangle} \langle x, \partial_v \rangle - |x|^2 (m + \mathbb{E}_x + \mathbb{E}_v - 2)(\mathbb{E}_u - \mathbb{E}_v) \\
&\quad - 2\langle u, x \rangle \langle x, \partial_u \rangle - 2\langle v, x \rangle \langle x, \partial_v \rangle \\
&\quad - I_Q \langle v, \partial_x \rangle I_Q \langle x, \partial_v \rangle - I_Q \langle u, \partial_x \rangle I_Q \langle x, \partial_u \rangle \\
[\langle x, \partial_v \rangle, I_Q \langle u, \partial_x \rangle I_Q] &= 2\langle u, x \rangle \langle x, \partial_v \rangle \\
[\langle x, \partial_u \rangle, I_Q \langle u, \partial_x \rangle I_Q] &= 2\langle u, x \rangle \langle x, \partial_u \rangle - |x|^2 (m + \mathbb{E}_x + \mathbb{E}_u - 1) \\
[\widetilde{\langle x, \partial_u \rangle}, I_Q \widetilde{\langle v, \partial_x \rangle} I_Q] &= 2\widetilde{\langle v, x \rangle} \widetilde{\langle x, \partial_u \rangle} - |x|^2 \langle v, \partial_u \rangle (\mathbb{E}_u - \mathbb{E}_v) \\
[\langle x, \partial_u \rangle, I_Q \langle v, \partial_x \rangle I_Q] &= 2\langle v, x \rangle \langle x, \partial_u \rangle - |x|^2 \langle v, \partial_u \rangle.
\end{aligned}$$

Proof. This follows from Lemma 42, Lemma 53 and Lemma 54. $\square$

10.3 A formulation of embedding factors

We define two important operators:

Definition 11. *The operators*

$$\Phi_u := \pi_1 I_Q \partial_x I_Q u$$
$$\Phi_v := \pi_1 I_Q \partial_x I_Q \widetilde{v}$$

are endomorphisms on the space of functions with values in the simplicial monogenics in two variables.

To see that these operators are well defined, consider a polynomial $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$. This implies that $uf \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k+1,l} \otimes \mathbb{S})$. We know that $I_Q \partial_x I_Q$ is an endomorphism of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k+1,l} \otimes \mathbb{S})$. Hence, the action with the projection operator π_1 makes sense. In short, if $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$, then $\Phi_u f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k+1,l})$ and, analogously, $\Phi_v f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l+1})$.

What we want to prove, is that

Conjecture 1. *One has*

$$\Phi_u : \mathrm{Ker}_h \mathcal{Q}_{k,l} \to \mathrm{Ker}_{h+1} \mathcal{Q}_{k+1,l}$$
$$\Phi_v : \mathrm{Ker}_h \mathcal{Q}_{k,l} \to \mathrm{Ker}_{h+1} \mathcal{Q}_{k,l+l}.$$

Status of the proof. So far, we can only state that

$$\Phi_u : \mathrm{Ker}_h \mathcal{Q}_{k,l} \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k+1,l})$$

which is visualised in

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \arrow[r, "I_Q \partial_x I_Q u"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k+1,l} \otimes \mathbb{S}) \arrow[d, "\pi_1"] \\
\mathrm{Ker}_h \mathcal{Q}_{k,l} \arrow[u, phantom, "\cup"] \arrow[r, dashed, "\Phi_u"'] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k+1,l})
\end{tikzcd}$$

and

$$\Phi_v : \mathrm{Ker}_h \mathcal{Q}_{k,l} \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l+l})$$

which is visualised in

$$\begin{tikzcd}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \arrow[r, "I_Q\partial_x I_Q\widetilde{v}"] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l+1} \otimes \mathbb{S}) \arrow[d, "\pi_1"] \\
\mathrm{Ker}_h\,\mathcal{Q}_{k,l} \arrow[u, phantom, "\cup"] \arrow[r, dashed, "\Phi_v"'] & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l+1})
\end{tikzcd}$$

At the time of writing, we are lost in calculations. □

Remark 41 (On proving the conjecture)**.** *Peter Van Lancker's approach [71] predicts that the weightless embedding factors (that translate to the operators Φ_u and Φ_v in the case of $\mathcal{Q}_{k,l}$) commute with the weightless higher spin Dirac operator. The underlying operator algebra could be related to transvector algebras and Yangians ([74, 54]).*

Even though calculations for the general case explode quickly, we have proved the conjecture for several examples. In order to find an explicit expression for Φ_u and Φ_v, consider again the operators

$$\begin{aligned}
I_Q\partial_x I_Q u &: \mathrm{Ker}_h\,\mathcal{Q}_{k,l} \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k+1,l} \otimes \mathbb{S})\\
I_Q\partial_x I_Q \widetilde{v} &: \mathrm{Ker}_h\,\mathcal{Q}_{k,l} \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l+l} \otimes \mathbb{S}).
\end{aligned}$$

The next proposition reveals that the operator $I_Q\partial_x I_Q u$ (resp. $I_Q\partial_x I_Q\widetilde{v}$), acting on null solutions of $\mathcal{Q}_{k,l}$, only belongs to two summands in the decomposition of $\mathcal{H}_{k+1,l} \otimes \mathbb{S}$ (resp. $\mathcal{H}_{k,l+1} \otimes \mathbb{S}$): the first one and the last one, where we use the following ordering of the summands:

$$\mathcal{H}_{k+1,l} \otimes \mathbb{S} = \underset{\uparrow}{\mathcal{S}_{k+1,l}} \oplus \widetilde{v}\mathcal{S}_{k+1,l-1} \oplus u\mathcal{S}_{k,l} \oplus \underset{\uparrow}{\langle \widetilde{u,v} \rangle \mathcal{S}_{k,l-1}}. \tag{10.10}$$

Explicitly,

$$\begin{aligned}
I_Q\partial_x I_Q u &: \mathrm{Ker}_h\,\mathcal{Q}_{k,l} \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k+1,l}) \oplus \mathcal{C}^\infty(\mathbb{R}^m, \langle \widetilde{u,v} \rangle \mathcal{S}_{k,l-1})\\
I_Q\partial_x I_Q \widetilde{v} &: \mathrm{Ker}_h\,\mathcal{Q}_{k,l} \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l+1}) \oplus \mathcal{C}^\infty(\mathbb{R}^m, \langle \widetilde{u,v} \rangle \mathcal{S}_{k-1,l}).
\end{aligned}$$

This is good news, because it reduces the complexity of the calculations. The proof goes as follows.

Proposition 27. *For a polynomial $f \in \mathrm{Ker}_h\,\mathcal{Q}_{k,l}$, one has*

(*i*) $\Phi_u f = I_Q\partial_x I_Q u f - \langle \widetilde{u,v} \rangle \pi_4 I_Q\partial_x I_Q u f$

(*ii*) $\Phi_v f = I_Q \partial_x I_Q \widetilde{v} f - \langle \widetilde{u, v} \rangle \pi_4 I_Q \partial_x I_Q \widetilde{v} f$.

Proof. (*i*) It follows from Lemma 42 that

$$I_Q \partial_x I_Q u f = 2 I_Q \langle u, \partial_x \rangle I_Q f - u I_Q \partial_x I_Q f.$$

Making use of Lemma 44, the action with the operators ∂_v, ∂_u, $\partial_u \partial_v$ and $\partial_v \partial_u$ on the above expression leads to

$$\begin{aligned}
\partial_v I_Q \partial_x I_Q u f &= u \partial_v I_Q \partial_x I_Q f \\
\partial_u I_Q \partial_x I_Q u f &= (m + 2\mathbb{E}_u - 2 + u\partial_u)\, I_Q \partial_x I_Q f \\
\partial_u \partial_v I_Q \partial_x I_Q u f &= -(m + 2\mathbb{E}_u) \partial_v I_Q \partial_x I_Q f \\
\partial_v \partial_u I_Q \partial_x I_Q u f &= (m + 2\mathbb{E}_u) \partial_v I_Q \partial_x I_Q f,
\end{aligned}$$

respectively. We can verify these results by using them to confirm that, indeed, $I_Q \partial_x I_Q u f \in \mathcal{H}_{k+1,l} \otimes \mathbb{S}$. It immediately follows from the two last equations that $\langle \partial_u, \partial_v \rangle I_Q \partial_x I_Q u f = 0$. Similarly, one can easily verify that $\Delta_u I_Q \partial_x I_Q u f = \Delta_v I_Q \partial_x I_Q u f = 0$.

Using the expression (6.8), the projection on the second summand in the decomposition of $\mathcal{H}_{k+1,l} \otimes \mathbb{S}$, which is the module $\mathcal{S}_{k+1,l-1}$, is now directly found:

$$\pi_2 I_Q \partial_x I_Q u f = \left(u\partial_v + u(m + 2\mathbb{E}_u)^{-1} \partial_u \partial_v \right) I_Q \partial_x I_Q u f = 0.$$

It follows from (6.9) that projection on $\mathcal{S}_{k,l} \hookrightarrow \mathcal{H}_{k+1,l} \otimes \mathbb{S}$, which is the third summand in the decomposition of this tensor product, leads to

$$\begin{aligned}
\pi_3 I_Q \partial_x I_Q u f &= \Big(m + 2\mathbb{E}_u - 2 + u\partial_u + (m + 2\mathbb{E}_v - 4)^{-1} \\
&\qquad \cdot \big(v(m + 2\mathbb{E}_u - 2) - 2u\langle v, \partial_u \rangle \big) \partial_v \Big) I_Q \partial_x I_Q u f \\
&= (m + 2\mathbb{E}_u - 2) I_Q \mathcal{Q}_{k,l} I_Q f = 0.
\end{aligned}$$

Note that, in this case, we actually had to use that $f \in \mathrm{Ker}_h \mathcal{Q}_{k,l}$. Finally, we have from (6.7) that projection on the summand $\mathcal{S}_{k,l-1}$ equals

$$\pi_4 I_Q \partial_x I_Q u f = -(m + 2\mathbb{E}_v - 2)^{-1} (m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1} \partial_v I_Q \partial_x I_Q f.$$

The first statement then follows from (6.10).

(*ii*) The proof of the second statement goes in a similar way. By means of Lemma 42, we have

$$I_Q\partial_x I_Q\widetilde{v}f = 2I_Q\widetilde{\langle v,\partial_x\rangle}I_Q f - \widetilde{v}I_Q\partial_x I_Q f$$

and it follows from Lemma 44 that the action with the operators ∂_v, ∂_u and $\partial_u\partial_v$ leads to

$$\begin{aligned}
\partial_v I_Q\partial_x I_Q\widetilde{v}f =& -2(\mathbb{E}_u-\mathbb{E}_v)I_Q\partial_x I_Q f - \{\partial_v,\widetilde{v}\}I_Q\partial_x I_Q f + \widetilde{v}\partial_v I_Q\partial_x I_Q f\\
=& \Big((m+2\mathbb{E}_v-4)(\mathbb{E}_u-\mathbb{E}_v) - u\partial_u + v\partial_v(\mathbb{E}_u-\mathbb{E}_v)\\
&- u\langle v,\partial_u\rangle\partial_v\Big)I_Q\partial_x I_Q f\\
\partial_u I_Q\partial_x I_Q\widetilde{v}f =& \Big(-\langle v,\partial_u\rangle(m+2\mathbb{E}_v-4) + v\partial_u(\mathbb{E}_u-\mathbb{E}_v)\\
&- u\langle v,\partial_u\rangle\partial_u\Big)I_Q\partial_x I_Q f\\
\partial_u\partial_v I_Q\partial_x I_Q\widetilde{v}f =& \ \partial_u(m+2\mathbb{E}_v-2)(\mathbb{E}_u-\mathbb{E}_v+1)I_Q\partial_x I_Q f\\
&+ \langle v,\partial_u\rangle\partial_v(m+2\mathbb{E}_v-2)I_Q\partial_x I_Q f,
\end{aligned}$$

respectively. Using the above expressions and (6.8), projection on the summand $\mathcal{S}_{k,l} \hookrightarrow \mathcal{H}_{k,l+1}\otimes\mathbb{S}$ leads to

$$\pi_2 I_Q\partial_x I_Q\widetilde{v}f = (m+2\mathbb{E}_v-4)(\mathbb{E}_u-\mathbb{E}_v)\pi_1 I_Q\partial_x I_Q f = 0.$$

In the same way, we have $\pi_3 I_Q\partial_x I_Q\widetilde{v}f = 0$, where we used once again that $\mathcal{Q}_{k,l}f = 0$. Finally,

$$\begin{aligned}
&\pi_4 I_Q\partial_x I_Q\widetilde{v}f\\
&= (m+2\mathbb{E}_u)^{-1}(m+\mathbb{E}_u+\mathbb{E}_v-1)^{-1}\Big(\partial_u(\mathbb{E}_u-\mathbb{E}_v+1)+\langle v,\partial_u\rangle\partial_v\Big)I_Q\partial_x I_Q f\\
&= (m+2\mathbb{E}_u)^{-1}(m+\mathbb{E}_u+\mathbb{E}_v-1)^{-1}\widetilde{\partial_u}I_Q\partial_x I_Q f,
\end{aligned}$$

which completes the proof. □

Remark 42. *We will show in Lemma 57 of section 10.4 that, in the special case that $f \in \mathrm{Ker}_h\mathcal{R}_k$, the first result (i) in this proposition reduces to*

$$\Phi_u f = I_Q\partial_x I_Q f.$$

This makes sense, because there exists no projection operator π_4 in the Rarita-Schwinger case.

Remark 43. *As we have been using Euler operators of u and v instead of explicit values, it is convenient to omit these homogeneity degrees and write the higher spin operator, usually denoted by $\mathcal{Q}_{k,l}$, as $\mathcal{Q}$ for short. Similarly, we write $\mathcal{R}$ instead of the usual $\mathcal{R}_k$. In [37], this 'weightless' approach is taken to the next level: the projection operators π_i are not expressed in terms of Euler operators, but by means of the Scasimir operator (see [3]).*

The statements of the above proposition can be written more explicitly.

Corollary 8. *One has*

$$
\begin{aligned}
\Phi_u \mathrm{Ker}\mathcal{Q} = {} & I_Q \partial_x I_Q u \, \mathrm{Ker}\mathcal{Q} \\
& + 2\widetilde{\langle u, v\rangle}(m + 2\mathbb{E}_v - 2)^{-1}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1} I_Q \langle \partial_v, \partial_x \rangle I_Q \mathrm{Ker}\mathcal{Q} \\
\Phi_v \mathrm{Ker}\mathcal{Q} = {} & I_Q \partial_x I_Q \widetilde{v} \, \mathrm{Ker}\mathcal{Q} \\
& - 2\widetilde{\langle u, v\rangle}(m + 2\mathbb{E}_u)^{-1}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1} I_Q \widetilde{\langle \partial_u, \partial_x \rangle} I_Q \mathrm{Ker}\mathcal{Q}.
\end{aligned}
$$

This is an interesting expression as it features the important operators $I_Q \partial_x I_Q$, $I_Q \widetilde{\langle \partial_u, \partial_x \rangle} I_Q$ and $I_Q \langle \partial_v, \partial_x \rangle I_Q$, together with the well-known embedding maps u, $\widetilde{v}$ and $\widetilde{\langle u, v\rangle}$. To emphasise the importance of these operators, we have the following result.

Lemma 56. *Acting on polynomials with values in the simplicial monogenics in two variables, one has*

$$
\begin{aligned}
I_Q \mathcal{Q} I_Q &= |x|^2 \mathcal{Q} \\
I_Q \widetilde{\langle \partial_u, \partial_x \rangle} I_Q &= \frac{1}{2}\{\widetilde{\partial_u}, I_Q \partial_x I_Q\} = -|x|^2 \widetilde{\langle \partial_u, \partial_x \rangle} + \widetilde{\langle x, \partial_u \rangle}(m + 2\mathbb{E}_u - 2) \\
I_Q \langle \partial_v, \partial_x \rangle I_Q &= \frac{1}{2}\{\partial_v, I_Q \partial_x I_Q\} = -|x|^2 \langle \partial_v, \partial_x \rangle + \langle x, \partial_v \rangle (m + 2\mathbb{E}_v - 4).
\end{aligned}
$$

By means of these results, the statements in Corollary 8 can be written even more explicitly.

Corollary 9. *Acting on* Ker$\mathcal{Q}$, *one has*

$$\begin{aligned}\Phi_u &= 2I_Q\langle u, \partial_x\rangle I_Q\; - u|x|^2\partial_x - 2|u|^2\widetilde{\langle x, \partial_u\rangle}(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1}\\ &\quad + 2u\widetilde{v}\langle x, \partial_v\rangle(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} + 2\widetilde{\langle u, v\rangle}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1}\langle x, \partial_v\rangle\\ &\quad - 2|x|^2\widetilde{\langle u, v\rangle}(m + 2\mathbb{E}_v - 2)^{-1}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1}\langle\partial_v, \partial_x\rangle\\ \Phi_v &= 2I_Q\widetilde{\langle v, \partial_x\rangle}I_Q\; - \widetilde{v}|x|^2\partial_x + 2\widetilde{v}u\widetilde{\langle x, \partial_u\rangle}(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1}\\ &\quad - 2|\widetilde{v}|^2\langle x, \partial_v\rangle(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} - 2\widetilde{\langle u, v\rangle}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1}\widetilde{\langle x, \partial_u\rangle}\\ &\quad - 2|x|^2\widetilde{\langle u, v\rangle}(m + 2\mathbb{E}_v - 2)^{-1}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1}\widetilde{\langle\partial_u, \partial_x\rangle}.\end{aligned}$$

Proof. This follows from Lemma 26, Lemma 42 and Lemma 56. □

Questions

To end this section, we list the problems that we would like to solve.

1. Proving Conjecture 1. Do we have, indeed, that

$$\begin{aligned}\Phi_u &: \text{Ker}\,\mathcal{Q} \to \text{Ker}\,\mathcal{Q}\\ \Phi_v &: \text{Ker}\,\mathcal{Q} \to \text{Ker}\,\mathcal{Q}\,?\end{aligned}$$

   This problem has been solved in the case of $l = 0$, as we show in the next section that

$$\Phi_u : \text{Ker}\,\mathcal{R} \to \text{Ker}\,\mathcal{R}.$$

2. Are these two operators commuting, i.e.

$$[\Phi_u, \Phi_v]\text{Ker}\,\mathcal{Q} = 0\,?$$

Remark 44. *Unlike the Rarita-Schwinger case in chapter 4 and [18], the operator* $I_Q\Delta_x I_Q\Delta_x$ *(see Lemma 12) is not a good choice of embedding factor for solutions of* $\mathcal{Q}_{k,l}$ *with* $l > 0$. *The reason for this is the presence of summands with multiplicity higher than one in* $\text{Ker}_h\mathcal{Q}_{k,l}$. *We illustrate this with an example. In* $\text{Ker}_h\mathcal{Q}_{2,1}$ *the irreducible summand* $\mathcal{S}_{h,1}$ *has multiplicity two. On the one hand, we have*

$$\begin{aligned}\text{Ker}_h\mathcal{Q}_{2,1} \cong\; &(h,2,1)' \oplus (h+1,1,1)' \oplus (h+1,2)' \oplus (h+2,1)' \oplus (h-2,1)'\\ &\oplus (h-1,1,1)' \oplus \underbrace{2(h,1)'} \oplus (h+1)' \oplus (h-1)' \oplus (h-1,2)'\end{aligned}$$

and, on the other hand,

$$\mathrm{Ker}_{h-2}\mathcal{Q}_{2,1} \cong (h-2,2,1)' \oplus (h-1,1,1)' \oplus (h-1,2)' \oplus \underbrace{(h,1)'} \oplus (h-4,1)'$$
$$\oplus (h-3,1,1)' \oplus 2(h-2,1)' \oplus (h-1)' \oplus (h-3)' \oplus (h-3,2)'.$$

Because we have that

$$\Delta_x : \mathrm{Ker}_h\mathcal{Q}_{2,1} \to \mathrm{Ker}_{h-2}\mathcal{Q}_{2,1},$$

the summand with multiplicity two is mapped to a summand with multiplicity one, which indicates that $I_Q\Delta_x I_Q$ *is not a good approach to find two linearly independent embedding factors. Indeed, we have that* $\widetilde{\langle v, \partial_x\rangle}\mathcal{S}_{h,1} \subset \mathrm{Ker}_{h-1}\mathcal{Q}_{1,1}$ *and* $\langle u, \partial_x\rangle\mathcal{S}_{h,1} \subset \mathrm{Ker}_{h-1}\mathcal{R}_1$*. Let* $g \in \mathcal{S}_{h,1}$*. We know from chapter 4 that we can invert*

$$\partial_x f = u\widetilde{\langle v, \partial_x\rangle} g$$

by means of

$$a_{hkl} f = I_Q\Delta_x I_Q\Delta_x f = -I_Q\Delta_x I_Q\partial_x u\widetilde{\langle v, \partial_x\rangle} g$$

for some normalisation constant a_{hkl}*. We can write this as*

$$f = \frac{1}{a_{hkl}} 2I_Q\Delta_x I_Q\langle u, \partial_x\rangle\widetilde{\langle v, \partial_x\rangle} g.$$

Because $[\langle u, \partial_x\rangle, \widetilde{\langle v, \partial_x\rangle}] = 0$*, inverting*

$$\partial_x f = \widetilde{v}\langle u, \partial_x\rangle g$$

leads to the same result. Hence, this approach leads to one embedding factor, which does not give the result we were hoping for. See also Lemma 63 in section 10.6.

10.4 Embedding factors in the R-S case

In [18] (see also section 4), the embedding factors in the Rarita-Schwinger case were constructed by means of the operator $I_R\Delta_x I_R\partial_x u$. An alternative embedding factor is given by the operator Φ_u in Definition 11. As mentioned in Remark 42, this operator reduces to the following expression.

Lemma 57. *One has*

$$\Phi_u\mathrm{Ker}\mathcal{R} = I_R\partial_x I_R u\mathrm{Ker}\mathcal{R}.$$

Proof. Starting from the left-hand side of the above expression, we write

$$\Phi_u \mathrm{Ker}\mathcal{R} = \pi_1 I_R \partial_x I_R u \mathrm{Ker}\mathcal{R} = \big(1 + (m + 2\mathbb{E}_u - 2)^{-1} u \, \underbrace{\partial_u\big) I_R \partial_x I_R u \mathrm{Ker}\mathcal{R}}\,.$$

Calculating the underlined expression leads to

$$\begin{aligned}
\partial_u I_R \partial_x I_R u \,\mathrm{Ker}\mathcal{R} &= [\partial_u, I_R \partial_x I_R u]\mathrm{Ker}\mathcal{R} \\
&= \{\partial_u, I_R \partial_x I_R\} u \,\mathrm{Ker}\mathcal{R} - I_R \partial_x I_R \{\partial_u, u\}\mathrm{Ker}\mathcal{R} \\
&= \big(m + 2\mathbb{E}_u - 2 + u\partial_u\big) I_R \partial_x I_R \mathrm{Ker}\mathcal{R} \\
&= (m + 2\mathbb{E}_u - 2)^{-1} I_R \mathcal{R} I_R \mathrm{Ker}\mathcal{R} \\
&= 0.
\end{aligned}$$

This means that

$$\Phi_u \mathrm{Ker}\mathcal{R} = \pi_1 I_R \partial_x I_R u \mathrm{Ker}\mathcal{R} = I_R \partial_x I_R u \mathrm{Ker}\mathcal{R},$$

which was also to be expected from Proposition 27, as there is no projection π_4 this time. □

Proposition 28. *For all integers $h > k > 0$, one has*

$$\Phi_u : \mathrm{Ker}_{h-1}\mathcal{R}_k \to \mathrm{Ker}_h \mathcal{R}_{k+1}.$$

Proof. The proof goes by induction on k, which is the degree of homogeneity in u of the null solutions of $\mathcal{R}_k$. We want to prove that

$$\mathcal{R}_{k+1} \Phi_u \mathrm{Ker}\mathcal{R}_k = \pi_1 \partial_x \Phi_u \mathrm{Ker}\mathcal{R}_k = 0.$$

Recall from the previous lemma that

$$\Phi_u \mathrm{Ker}\mathcal{R}_k = I_R \partial_x I_R u \mathrm{Ker}\mathcal{R}_k.$$

By means of results in Lemma 43, we have for $k = 0$:

$$\begin{aligned}
\pi_1 \partial_x \left(I_R \partial_x I_R u\right) \mathcal{M}_{h-1} &= 2\pi_1 \partial_x I_R \langle u, \partial_x \rangle I_R \mathcal{M}_{h-1} \\
&= 2\pi_1 [\partial_x, I_R \langle u, \partial_x \rangle I_R] \mathcal{M}_{h-1} \\
&= -4(m + h - 2)\pi_1 u \mathcal{M}_{h-1} = 0.
\end{aligned}$$

Hence,

$$\Phi_u : \mathrm{Ker}_{h-1}\partial_x \to \mathrm{Ker}_h \mathcal{R}_1$$

and

$$\begin{aligned}\mathrm{Ker}_h\mathcal{R}_1 &= \mathcal{S}_{h,1} \oplus \langle u, \partial_x\rangle \mathcal{M}_{h+1} \oplus \Phi_u \mathcal{M}_{h-1}\\ &= \bigoplus_{i=0}^{1}\bigoplus_{j=0}^{1-i}(\Phi_u)^i\langle u,\partial_x\rangle^j \mathcal{S}_{h+j-i,1-i-j}.\end{aligned}$$

Now, we state that every element in $\mathrm{Ker}_{h-1}\mathcal{R}_{k-1}$ is of the form

$$\mathrm{Ker}_h\mathcal{R}_{k-1} = \bigoplus_{i=0}^{k-1}\bigoplus_{j=0}^{k-1-i}(\Phi_u)^i\langle u,\partial_x\rangle^j \mathcal{S}_{h-1+j-i,k-1-i-j}.$$

Assuming this induction hypothesis to hold, we can also write

$$f = \Phi_u \underbrace{(\Phi_u)^{i-1}\langle u,\partial_x\rangle^j \mathcal{S}_{h-1+j-(i-1),k-1-(i-1)-j}}_{\substack{\Cap \\ \mathrm{Ker}_{h-1}\mathcal{R}_{k-1}}} \tag{10.11}$$

for i, j satisfying $i + j \in [0, k]$. We will now prove that $f \in \mathrm{Ker}_h\mathcal{R}_k$, or

$$\pi_1\partial_x f = 0.$$

By means of $I_R\partial_x I_R u = 2I_R\langle u,\partial_x\rangle I_R - uI_R\partial_x I_R$, (10.11) can be written as

$$f = \sum_{t=0}^{j}(-1)^t 2^{i-t}\left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t}\begin{pmatrix} i \\ t\end{pmatrix}\underbrace{(uI_R\partial_x I_R)^t\langle u,\partial_x\rangle^j \mathcal{S}_{h+j-i,k-i-j}}.$$

Let us deal with the underlined expression above. For a function g with values in the spherical monogenics, it is not difficult to calculate that

$$\begin{aligned}(uI_R\partial_x I_R)\langle u,\partial_x\rangle^j g &= u[I_R\partial_x I_R, \langle u,\partial_x\rangle^j]g\\ &= -2ju^2\langle u,\partial_x\rangle^{j-1}(\mathbb{E}_x - \mathbb{E}_u - j + 1)g.\end{aligned}$$

By means of this result, we have

$$(uI_R\partial_x I_R)^t\langle u,\partial_x\rangle^j g = (-2)^t u^{2t}\frac{j!}{(j-t)!}\langle u,\partial_x\rangle^{j-t}\frac{(\mathbb{E}_x - \mathbb{E}_u - j + t)!}{(\mathbb{E}_x - \mathbb{E}_u - j)!}g$$

and we can rewrite (10.11) as

$$f = 2^i\left(\sum_{t=0}^{j} C_t u^{2t}\left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t}\langle u,\partial_x\rangle^{j-t}\right)\mathcal{S}_{h+j-i,k-i-j}$$

with
$$C_t := \begin{pmatrix} i \\ t \end{pmatrix} \frac{j!}{(j-t)!} \frac{(h-k+j+t)!}{(h-k+j)!}.$$
This constant satisfies
$$(i-t+1)(j-t+1)(h-k+j+t)C_{t-1} = tC_t. \tag{10.12}$$
Finally, we calculate $\partial_x f$:
$$\begin{aligned}\partial_x f \;=\; & 2^i \Bigg(\sum_{t=0}^{j} C_t u^{2t} [\partial_x, (I_R\langle u,\partial_x\rangle I_R)^{i-t}] \langle u,\partial_x\rangle^{j-t} \Bigg) \mathcal{S}_{h+j-i,k-i-j} \\ & + 2^i \Bigg(\sum_{t=0}^{j} C_t u^{2t} \left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t} [\partial_x, \langle u,\partial_x\rangle^{j-t}] \Bigg) \mathcal{S}_{h+j-i,k-i-j}.\end{aligned}$$
The right-hand side can be written as
$$\begin{aligned}2^i & \Bigg(\sum_{t=0}^{j} (-2u) C_t u^{2t} (i-t)(m+h-k+t-2) \left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t-1} \langle u,\partial_x\rangle^{j-t} \\ & + \sum_{t=0}^{j-1} 2u^3 C_t u^{2t} (i-t)(i-t-1)(j-t)(h-k+j+t+1) \\ & \cdot \left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t-2} \langle u,\partial_x\rangle^{j-t-1} \Bigg) \mathcal{S}_{h+j-i,k-i-j}.\end{aligned}$$
Changing the summation indices in the second term and using (10.12), we find
$$\begin{aligned}2^i & \Bigg(\sum_{t=0}^{j} (-2) C_t u^{2t+1} (i-t)(m+h-k+t-2) \left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t-1} \langle u,\partial_x\rangle^{j-t} \\ & + \sum_{t=0}^{j} 2t C_t u^{2t+1} (i-t) \left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t-1} \langle u,\partial_x\rangle^{j-t} \Bigg) \mathcal{S}_{h+j-i,k-i-j} \\ = & -2^{i+1} D_{h,k} u \left(\sum_{t=0}^{j} C_t u^{2t} (i-t) \left(I_R\langle u,\partial_x\rangle I_R\right)^{i-t-1} \langle u,\partial_x\rangle^{j-t} \right) \mathcal{S}_{h+j-i,k-i-j}\end{aligned}$$
with $D_{h,k} := m+h-k-2$. Rewriting once more the right-hand side, we conclude
$$\partial_x f \;=\; -4iD_{h,k} u (\Phi_u)^{i-1} \langle u,\partial_x\rangle^j \mathcal{S}_{h+j-i,k-i-j}$$

(recall that i is an integer in $\{0, \ldots, k\}$ and is not to be confused with the complex unit). Hence, it immediately follows that

$$\mathcal{R}_k f = \pi_1 \partial_x f = 0,$$

which concludes the proof. □

10.5 Embedding factors for type A solutions

In this section, we investigate the embedding map of two vector spaces of null solutions in $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ (with $k > l$): the type A solutions from $\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1}$ and $\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l}$, which are given by $\mathcal{M}^s_{h-1,k,l-1}$ and $\mathcal{M}^s_{h-1,k-1,l}$, respectively. By means of the next two lemmas, we will prove that, once again, Φ_u and Φ_v are good choices.

Lemma 58. *For all integers $p \geq q \geq r$, one has*

$$I_Q \partial_x I_Q \mathcal{M}^s_{p,q,r} \cong u\mathcal{M}^s_{p+1,q-1,r} \oplus \widetilde{v}\mathcal{M}^s_{p+1,q,r-1}.$$

Proof. It follows from (10.4), i.e.

$$\begin{aligned} I_Q \partial_x I_Q &= |x|^2 \partial_x + 2\langle x, u\rangle \partial_u + 2\langle x, v\rangle \partial_v \\ &\quad - 2u\widetilde{\langle x, \partial_u\rangle}(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1} - 2\widetilde{v}\langle x, \partial_v\rangle(\mathbb{E}_u - \mathbb{E}_v + 1)^{-1}, \end{aligned}$$

that

$$I_Q \partial_x I_Q \mathcal{M}^s_{p,q,r} = -2(q - r + 1)^{-1}\left(u\widetilde{\langle x, \partial_u\rangle} + \widetilde{v}\langle x, \partial_v\rangle\right)\mathcal{M}^s_{p,q,r}.$$

By means of Lemma 52, we have

$$\begin{aligned} \widetilde{\langle x, \partial_u\rangle}\mathcal{M}^s_{p,q,r} &= \bigoplus_{i=0}^{q-r}\bigoplus_{j=0}^{r} \widetilde{\langle v, \partial_x\rangle}^j [\widetilde{\langle x, \partial_u\rangle}, \langle u, \partial_x\rangle^i]\mathcal{S}_{p+i+j,q-i,r-j} \\ &= \bigoplus_{i=1}^{q-r}\bigoplus_{j=0}^{r} \alpha_i \langle u, \partial_x\rangle^{i-1}\widetilde{\langle v, \partial_x\rangle}^j \mathcal{S}_{p+i+j,q-i,r-j} \end{aligned}$$

with

$$\alpha_i = i(p - q + j + \frac{3i}{2} + \frac{1}{2})(q - r + j - 1).$$

Once again, i is an integer in the set $\{1, 2, \ldots, q-r\}$, not to be confused with the complex unit. Note that the above expression can be written as

$$\begin{aligned}\langle\widetilde{x, \partial_u}\rangle\mathcal{M}^s_{p,q,r} &= \bigoplus_{i=0}^{q-r-1}\bigoplus_{j=0}^{r}(i+1)\alpha_{i+1}\langle u, \partial_x\rangle^i\langle\widetilde{v, \partial_x}\rangle^j\mathcal{S}_{p+1+i+j,q-1-i,r-j}\\ &\cong \mathcal{M}^s_{p+1,q-1,r}.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle x, \partial_v\rangle\mathcal{M}^s_{p,q,r} &= \bigoplus_{i=0}^{q-r}\bigoplus_{j=0}^{r}\langle u, \partial_x\rangle^i[\langle x, \partial_v\rangle, \langle\widetilde{v, \partial_x}\rangle^j]\mathcal{S}_{p+i+j,q-i,r-j}\\ &= \bigoplus_{i=0}^{q-r}\bigoplus_{j=1}^{r}\beta_j\langle u, \partial_x\rangle^i\langle\widetilde{v, \partial_x}\rangle^{j-1}\mathcal{S}_{p+i+j,q-i,r-j}\end{aligned}$$

with

$$\beta_j = j\left(p - r + i + \frac{3j}{2} + \frac{3}{2}\right)(q - r - i + 1).$$

This can be written as

$$\begin{aligned}\langle\widetilde{x, \partial_u}\rangle\mathcal{M}^s_{p,q,r} &= \bigoplus_{i=0}^{q-}\bigoplus_{j=0}^{r-1}(j+1)\beta_{j+1}\langle u, \partial_x\rangle^i\langle\widetilde{v, \partial_x}\rangle^j\mathcal{S}_{p+1+i+j,q-i,r-1-j}\\ &\cong \mathcal{M}^s_{p+1,q,r-1},\end{aligned}$$

which concludes the proof. $\square$

In particular, it follows from Lemma 56 that

$$\begin{aligned}I_Q\langle\widetilde{\partial_u, \partial_x}\rangle I_Q\mathcal{M}^s_{p,q,r} &\cong \mathcal{M}^s_{p+1,q-1,r}\\ I_Q\langle\partial_v, \partial_x\rangle I_Q\mathcal{M}^s_{p,q,r} &\cong \mathcal{M}^s_{p+1,q,r-1}.\end{aligned}$$

By means of the previous lemma, we can prove that Φ_u and Φ_v are embedding factors for solutions of type A.

Lemma 59. *For all integers $p \geq q > r$, one has*

$$\begin{aligned}\Phi_u &: \mathcal{M}^s_{p,q,r} \rightarrow \mathrm{Ker}_{p+1}\mathcal{Q}_{q+1,r}\\ \Phi_v &: \mathcal{M}^s_{p,q,r} \rightarrow \mathrm{Ker}_{p+1}\mathcal{Q}_{q,r+1}.\end{aligned}$$

Proof. This is equivalent to

$$\pi_1 \partial_x \Phi_u \mathcal{M}^s_{p,q,r} = 0 \quad \text{and} \quad \pi_1 \partial_x \Phi_v \mathcal{M}^s_{p,q,r} = 0,$$

respectively. The proof of the first statement goes as follows. We have

$$\begin{aligned}\Phi_u \mathcal{M}^s_{p,q,r} &= 2I_Q \langle u, \partial_x \rangle I_Q \mathcal{M}^s_{p,q,r} - uI_Q \partial_x I_Q \mathcal{M}^s_{p,q,r} \\ &+ 2\widetilde{\langle u, v \rangle}(m + 2\mathbb{E}_v - 2)^{-1}(m + \mathbb{E}_u + \mathbb{E}_v - 1)^{-1}\widetilde{\langle u, v \rangle} I_Q \langle \partial_v, \partial_x \rangle I_Q \mathcal{M}^s_{p,q,r}\end{aligned}$$

and it follows from the previous lemma that

$$\begin{aligned}\Phi_u \mathcal{M}^s_{p,q,r} \cong\ & 2I_Q \langle u, \partial_x \rangle I_Q \mathcal{M}^s_{p,q,r} - u \left(u\mathcal{M}^s_{p+1,q-1,r} + \widetilde{v}\mathcal{M}^s_{p+1,q,r-1} \right) \\ &+ \widetilde{\langle u, v \rangle} \mathcal{M}^s_{p+1,q,r-1}.\end{aligned}$$

Then,

$$\begin{aligned}\partial_x \Phi_u \mathcal{M}^s_{p,q,r} &= [\partial_x, \Phi_u]\mathcal{M}^s_{p,q,r} \\ &\cong 2[\partial_x, I_Q \langle u, \partial_x \rangle I_Q]\mathcal{M}^s_{p,q,r} - [\partial_x, u\widetilde{v}]\mathcal{M}^s_{p+1,q,r-1} \\ &\quad + [\partial_x, \widetilde{\langle u, v \rangle}]\mathcal{M}^s_{p+1,q,r-1}.\end{aligned}$$

By means of Lemma 37 and Lemma 43, this result can be written as

$$\partial_x \Phi_u \mathcal{M}^s_{p,q,r} \cong u\mathcal{S}_1 + \widetilde{v}\mathcal{S}_2$$

with $\mathcal{S}_1 \subset \mathcal{P}_p(\mathbb{R}^m, \mathcal{S}_{q,r})$ and $\mathcal{S}_2 \subset \mathcal{P}_p(\mathbb{R}^m, \mathcal{S}_{q+1,r-1})$. Hence, after taking the projection π_1, we find

$$\pi_1 \partial_x \Phi_u \mathcal{M}^s_{p,q,r} = 0.$$

In a similar way, one proves the second statement. $\square$

We can conclude from this lemma that Conjecture 1 holds for the solutions of type A from $\text{Ker}_{h-1}\mathcal{Q}_{k-1,l}$ and $\text{Ker}_{h-1}\mathcal{Q}_{k,l-1}$. Theses spaces of null solutions can be embedded in the kernel space $\text{Ker}_h\mathcal{Q}_{k,l}$ with embedding factor Φ_u and Φ_v, respectively.

10.6 Embedding factors of $\text{Ker}_h\mathcal{Q}_{2,1}$

Recall that, in order to construct $\text{Ker}_h\mathcal{Q}_{2,1}$, the decomposition of three spaces of type B solutions was important. They are given by

1. $\operatorname{Ker}_{h-1}\mathcal{Q}_{1,1} \cap \operatorname{Ker}\langle \partial_v, \partial_x \rangle$
2. $\operatorname{Ker}_{h-1}\mathcal{R}_2 \cap \operatorname{Ker}\langle \partial_u, \partial_x \rangle$
3. $\operatorname{Ker}_{h-2}\mathcal{R}_1 \cap \operatorname{Im}\langle \partial_u, \partial_x \rangle \cap \operatorname{Im}\langle \partial_v, \partial_x \rangle$,

and are visualised in the following diagram:

$$\begin{tikzcd}
\operatorname{Ker}_{h-1}\mathcal{R}_2 \arrow[d, "\langle \partial_u, \partial_x \rangle"'] & \operatorname{Ker}_h \mathcal{Q}_{2,1} \arrow[l, "\langle \partial_v, \partial_x \rangle"'] \arrow[d, "\langle \widetilde{\partial_u, \partial_x} \rangle"] \\
\operatorname{Ker}_{h-2}\mathcal{R}_1 & \operatorname{Ker}_{h-1}\mathcal{Q}_{1,1} \arrow[l, "\langle \partial_v, \partial_x \rangle"]
\end{tikzcd}$$

We will investigate the embedding factors of the $\operatorname{Spin}(m)$-irreducible summands of each of these vector spaces.

1. $\mathbf{Ker}_{h-1}\mathcal{Q}_{1,1} \cap \mathbf{Ker}\langle \partial_v, \partial_x \rangle$

$$\begin{tikzcd}
\cdot \arrow[r, dash] \arrow[d, dash] & \operatorname{Ker}_h \mathcal{Q}_{2,1} \\
\cdot \arrow[r, dash] & \operatorname{Ker}_{h-1}\mathcal{Q}_{1,1} \cap \operatorname{Ker}\langle \partial_v, \partial_x \rangle \arrow[u, hook]
\end{tikzcd}$$

The space $\operatorname{Ker}_{h-1}\mathcal{Q}_{1,1} \cap \operatorname{Ker}\langle \partial_v, \partial_x \rangle$ is equal to $\mathcal{S}_{h-1,1,1} \oplus \langle \widetilde{v, \partial_x} \rangle \mathcal{S}_{h,1}$, which is the decomposition of $\mathcal{M}^s_{h-1,1,1}$. Both spaces can embedded in $\operatorname{Ker}_h \mathcal{Q}_{2,1}$ as follows:

Lemma 60. *One has*

(i) $\Phi_u \mathcal{S}_{h-1,1,1} \subset \operatorname{Ker}_h \mathcal{Q}_{2,1}$

(ii) $\Phi_u \langle \widetilde{v, \partial_x} \rangle \mathcal{S}_{h,1} \subset \operatorname{Ker}_h \mathcal{Q}_{2,1}$.

Proof. Although the above statements immediately follow from Lemma 59, which translates to

$$\Phi_u : \mathcal{M}^s_{h-1,1,1} \to \operatorname{Ker}_h \mathcal{Q}_{2,1},$$

it can be interesting to do the calculations explicitly in the hope to discover certain patterns. Unfortunately, these calculations get very complex, which indicates that a more general approach is necessary.

(i) We will show that $\mathcal{Q}_{2,1}\Phi_u\mathcal{S}_{h-1,1,1} = \pi_1\partial_x\Phi_u\mathcal{S}_{h-1,1,1} = 0$. Using Corollary 3 and 6, we obtain

$$\begin{aligned}\Phi_u\mathcal{S}_{h-1,1,1} &= \pi_1\{I_Q\partial_x I_Q, u\}\mathcal{S}_{h-1,1,1}\\ &= 2\pi_1 I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-1,1,1}\\ &= 2I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-1,1,1}.\end{aligned}$$

Acting with ∂_x leads to

$$\begin{aligned}\partial_x\Phi_u\mathcal{S}_{h-1,1,1} &= 2[\partial_x, I_Q\langle u,\partial_x\rangle I_Q]\mathcal{S}_{h-1,1,1}\\ &= -4(m+h-1)u\,\mathcal{S}_{h-1,1,1},\end{aligned}$$

from which it follows that $\mathcal{Q}_{2,1}\Phi_u\mathcal{S}_{h-1,1,1} = 0$.

(ii) Proving that $\mathcal{Q}_{2,1}\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1} = \pi_1\partial_x\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1} = 0$ is less straightforward than the previous case. Applying Corollary 8, we find

$$\begin{aligned}\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1} &= \pi_1 I_Q\partial_x I_Q u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}\\ &= I_Q\partial_x I_Q u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1} + 2\frac{(h+1)}{m}\langle\widetilde{u,v}\rangle\mathcal{S}_{h,1},\end{aligned}$$

which can be expanded to

$$\begin{aligned}&\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}\\ &= 2I_Q\langle u,\partial_x\rangle I_Q\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1} - u[I_Q\partial_x I_Q, \langle\widetilde{v,\partial_x}\rangle]\mathcal{S}_{h,1} + 2\frac{(h+1)}{m}\langle\widetilde{u,v}\rangle\mathcal{S}_{h,1}\\ &= \big(-2|x|^2\langle u,\partial_x\rangle - 2(m+2)\langle x,u\rangle - 2ux\big)\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}\\ &\quad + 4(h+1)|u|^2\langle v,\partial_u\rangle\mathcal{S}_{h,1} + 4(h+1)\langle u,v\rangle\mathcal{S}_{h,1}\\ &\quad + 2(h+1)u\widetilde{v}\mathcal{S}_{h,1} + 2\frac{(h+1)}{m}\langle\widetilde{u,v}\rangle\mathcal{S}_{h,1}\\ &= \big(-2|x|^2\langle u,\partial_x\rangle - 2(m+2)\langle x,u\rangle - 2ux\big)\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}\\ &\quad - 2\frac{(h+1)(m+1)}{m}|u|^2\langle v,\partial_u\rangle\mathcal{S}_{h,1} + 2(h+1)\langle v,u\rangle\mathcal{S}_{h,1} - 2\frac{(h+1)}{m}vu\mathcal{S}_{h,1}.\end{aligned} \tag{10.13}$$

Then

$$\begin{aligned}&\partial_x\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}\\ &\quad = -2\left(2m+2h-2-\frac{(h+1)}{m}\right)u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1} - 2\frac{(h+1)}{m}\widetilde{v}\langle u,\partial_x\rangle\mathcal{S}_{h,1},\end{aligned}$$

from which it follows that $\mathcal{Q}_{2,1}\Phi_u\langle \widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}=0$. $\square$

2. $\mathbf{Ker}_{h-1}\mathcal{R}_2\cap\mathbf{Ker}\langle\partial_u,\partial_x\rangle$

$$\begin{tikzcd} \mathrm{Ker}_{h-1}\mathcal{R}_2 \arrow[r, hook] \arrow[d, dash] & \mathrm{Ker}_h\mathcal{Q}_{2,1} \arrow[d, dash] \\ \cdot \arrow[r, dash] & \cdot \end{tikzcd}$$

Next, we deal with $\mathrm{Ker}_{h-1}\mathcal{R}_2\cap\mathrm{Ker}\langle\partial_u,\partial_x\rangle$, which consists of three pieces: $\mathcal{S}_{h-1,2,0}$, $\langle u,\partial_x\rangle\mathcal{S}_{h,1}$ and $\langle u,\partial_x\rangle^2\mathcal{S}_{h+1,0}$, and we have

$$\mathcal{M}^s_{h-1,2,0}=\mathcal{S}_{h-1,2,0}\oplus\langle u,\partial_x\rangle\mathcal{S}_{h,1}\oplus\langle u,\partial_x\rangle^2\mathcal{S}_{h+1,0}.$$

They can be embedded in $\mathrm{Ker}_h\mathcal{Q}_{2,1}$ as follows:

Lemma 61. *One has*

(i) $\Phi_v\mathcal{S}_{h-1,2,0}\subset\mathrm{Ker}_h\mathcal{Q}_{2,1}$

(ii) $\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}\subset\mathrm{Ker}_h\mathcal{Q}_{2,1}$

(iii) $\Phi_v\langle u,\partial_x\rangle^2\mathcal{S}_{h+1,0,0}\subset\mathrm{Ker}_h\mathcal{Q}_{2,1}$.

Proof. Even though these statements follow from Lemma 59, which translates to

$$\Phi_v:\mathcal{M}^s_{h-1,2,0}\to\mathrm{Ker}_h\mathcal{Q}_{2,1},$$

we proceed with explicit calculations, for the same reason as mentioned in the previous lemma.

(i) Proving that $\pi_1\partial_x\Phi_v\mathcal{S}_{h-1,2,0}=0$ is easy. We begin with

$$\begin{aligned}\Phi_v\mathcal{S}_{h-1,2,0} &= \pi_1\{I_Q\partial_xI_Q,\widetilde{v}\}\mathcal{S}_{h-1,2,0}\\ &= 2\pi_1I_Q\langle\widetilde{v,\partial_x}\rangle I_Q\mathcal{S}_{h-1,2,0}\\ &= 2I_Q\langle\widetilde{v,\partial_x}\rangle I_Q\mathcal{S}_{h-1,2,0}\end{aligned}$$

and action with ∂_x leads to

$$\begin{aligned}\partial_x\Phi_v\mathcal{S}_{h-1,2,0} &= 2[\partial_x,I_Q\langle\widetilde{v,\partial_x}\rangle I_Q]\mathcal{S}_{h-1,2,0}\\ &= -4(m+h-3)\widetilde{v}\,\mathcal{S}_{h-1,2,0},\end{aligned}$$

from which it follows that $\mathcal{Q}_{2,1}\Phi_v\mathcal{S}_{h-1,2,0}=0$. $\square$

(ii) It follows from Corollary 8 that

$$\begin{aligned}\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1} \;=\;& \left(-2|x|^2\widetilde{\langle v,\partial_x\rangle}-2(m-2)\widetilde{\langle x,v\rangle}-2\widetilde{v}x\right)\langle u,\partial_x\rangle\mathcal{S}_{h,1}\\ &-2\frac{(h-1)(m-3)}{m}|u|^2\langle v,\partial_u\rangle\mathcal{S}_{h,1}+2(h-1)\langle v,u\rangle\mathcal{S}_{h,1}\\ &+6\frac{(h-1)}{m}vu\mathcal{S}_{h,1}. \end{aligned} \tag{10.14}$$

Acting with ∂_x leads to

$$\begin{aligned}\partial_x\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1} =& -2\left(2m+2h-6+3\frac{(h-1)}{m}\right)\widetilde{v}\langle u,\partial_x\rangle\mathcal{S}_{h,1}\\ &-6\frac{(h-1)}{m}u\widetilde{\langle v,\partial_x\rangle}\mathcal{S}_{h,1},\end{aligned}$$

as a consequence of which

$$\pi_1\partial_x\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}=\mathcal{Q}_{2,1}\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}=0.$$

(iii) Because $\mathcal{S}_{h+1,0,0}=\mathcal{M}_{h+1}$ does not depend on u or v, the calculations in this case are not very complex. We begin by calculating

$$\begin{aligned}\Phi_v\langle u,\partial_x\rangle^2\mathcal{M}_{h+1} \;=\;& 2I_Q\widetilde{\langle v,\partial_x\rangle}I_Q\langle u,\partial_x\rangle^2\mathcal{M}_{h+1}+4h\widetilde{v}u\langle u,\partial_x\rangle\mathcal{M}_{h+1}\\ &-\frac{12h}{m}\left(m\langle u,v\rangle+(m-1)vu-|u|^2\langle v,\partial_u\rangle\right)\langle u,\partial_x\rangle\mathcal{M}_{h+1}.\end{aligned}$$

Acting with ∂_x leads to

$$\partial_x\Phi_v\langle u,\partial_x\rangle^2\mathcal{M}_{h+1} \;=\; -4\left(m+h-3-\frac{3h}{m}\right)\widetilde{v}\langle u,\partial_x\rangle^2\mathcal{M}_{h+1}$$

Hence, we have

$$\mathcal{Q}_{2,1}\Phi_v\langle u,\partial_x\rangle^2\mathcal{M}_{h+1}=\pi_1\partial_x\Phi_v\langle u,\partial_x\rangle^2\mathcal{M}_{h+1}=0.$$

□

The space $\mathcal{S}_{h,1}$ occurs twice in the decomposition of $\mathrm{Ker}_h\,\mathcal{Q}_{2,1}$: once with embedding factor $\Phi_u\widetilde{\langle v,\partial_x\rangle}$ and the other time with embedding factor $\Phi_v\langle u,\partial_x\rangle$. To ensure that $\mathcal{S}_{h,1}$ has multiplicity two in the decomposition of $\mathrm{Ker}_h\,\mathcal{Q}_{2,1}$, we prove that

Lemma 62. *The spaces* $\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}$ *and* $\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}$ *are linear independent.*

Proof. It follows from (10.13) and (10.14) that

$$\alpha\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}+\beta\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}=0\;\Rightarrow\;\alpha=\beta=0,$$

for constants α and β. $\square$

The following lemma shows that the operator $I_Q\Delta_x I_Q\langle u,\partial_x\rangle\langle\widetilde{v,\partial_x}\rangle$ (which is discussed in remark 44) gives rise to an embedding factor for $\mathcal{S}_{h,1}$ that is connected to Φ_u and Φ_v as follows:

Lemma 63. *One has*

$$I_Q\Delta_x I_Q\langle u,\partial_x\rangle\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}=-3(h-1)\Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}-(h+1)\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}.$$

Proof. This follows from a direct calculation using results of this section. $\square$

3. $\mathbf{Ker}_{h-2}\mathcal{R}_1\cap\mathbf{Im}\langle\widetilde{\partial_u,\partial_x}\rangle\cap\mathbf{Im}\langle\partial_v,\partial_x\rangle$

$$\begin{tikzcd} \cdot \arrow[r, dash] \arrow[d, dash] & \mathrm{Ker}_h\mathcal{Q}_{2,1} \arrow[d, dash] \\ \mathrm{Ker}_{h-2}\mathcal{R}_1 \arrow[r, dash] \arrow[ur, hook] & \cdot \end{tikzcd}$$

Finally, we will with the space $\mathrm{Ker}_{h-2}\mathcal{R}_1\cap\mathrm{Im}\langle\widetilde{\partial_u,\partial_x}\rangle\cap\mathrm{Im}\langle\partial_v,\partial_x\rangle$. This intersection consists of the vector spaces $\mathcal{S}_{h-2,1,0}$ and $\langle u,\partial_x\rangle\mathcal{S}_{h-1,0,0}$, which constitute the decomposition of $\mathcal{M}^s_{h-2,1,0}$ into Spin(m)-irreducibles.

Lemma 64. *One has*

(*i*) $\Phi_u\Phi_v\mathcal{S}_{h-2,1,0}=\Phi_u\Phi_v\mathcal{S}_{h-2,1,0}\subset\mathrm{Ker}_h\mathcal{Q}_{2,1}$

(*ii*) $\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h-1,0,0}=\Phi_v\Phi_u\langle u,\partial_x\rangle\mathcal{S}_{h-1,0,0}\subset\mathrm{Ker}_h\mathcal{Q}_{2,1}$.

Proof. On the one hand, we will prove that $[\Phi_u,\Phi_v]\mathcal{M}^s_{h-2,1,0}=0$, and on the other hand, we will show that $\Phi_u\Phi_v\mathcal{M}^s_{h-2,1,0}\subset\mathrm{Ker}_h\mathcal{Q}_{2,1}$.

(*i*) Because it does not have embedding factors, the space $\mathcal{S}_{h-2,1}$, i.e. the (first) summand in $\mathcal{M}^s_{h-2,1,0}$, is not difficult to embed. We have

$$\Phi_u\mathcal{S}_{h-2,1,0}\;=\;2I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0}$$

and

$$\begin{aligned}\Phi_v\Phi_u\mathcal{S}_{h-2,1,0} &= 4I_Q\langle \widetilde{v,\partial_x}\rangle I_Q I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0}\\ &= -4I_Q\langle \widetilde{v,\partial_x}\rangle\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0},\end{aligned}$$

which equals

$$\begin{aligned}\Phi_u\Phi_v\mathcal{S}_{h-2,1,0} &= 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle \widetilde{v,\partial_x}\rangle I_Q\mathcal{S}_{h-2,1,0}\\ &= -4I_Q\langle \widetilde{v,\partial_x}\rangle\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0}.\end{aligned}$$

By means of Lemma 43, acting with ∂_x leads to

$$\begin{aligned}4\partial_x I_Q\langle \widetilde{v,\partial_x}\rangle I_Q I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0} &= 4[\partial_x, I_Q\langle \widetilde{v,\partial_x}\rangle I_Q]I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0}\\ &\quad + 4I_Q\langle \widetilde{v,\partial_x}\rangle I_Q[\partial_x, I_Q\langle u,\partial_x\rangle I_Q]\mathcal{S}_{h-2,1,0}\\ &= -8(m+h-3)\widetilde{v}I_Q\langle u,\partial_x\rangle I_Q\mathcal{S}_{h-2,1,0}\\ &\quad - 8(m+h-2)uI_Q\langle \widetilde{v,\partial_x}\rangle I_Q\mathcal{S}_{h-2,1,0}\\ &= -4(m+h-3)\widetilde{v}\Phi_u\mathcal{S}_{h-2,1,0}\\ &\quad - 4(m+h-2)u\Phi_v\mathcal{S}_{h-2,1,0},\end{aligned}$$

whence $\mathcal{Q}_{2,1}\Phi_v\Phi_u\mathcal{S}_{h-2,1,0} = 0$.

(ii) Before proving that $\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h-1,0,0}$ belongs to $\mathrm{Ker}_h\mathcal{Q}_{2,1}$, we calculate the explicit expression of this space. It follows from Corollary 8 that

$$\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} = 2I_Q\langle \widetilde{v,\partial_x}\rangle I_Q\langle u,\partial_x\rangle\mathcal{M}_{h-1} - 4\left(\frac{h-1}{m-1}\right)u\wedge v\mathcal{M}_{h-1}.$$

Note that

$$I_Q\partial_x I_Q\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} = 4\frac{(h-1)(m-3)}{m-1}\widetilde{v}I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}.$$

The calculation of $\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1}$ goes in two steps. First, we consider the action with $I_Q\partial_x I_Q u$ on the above expression, then we project with π_1.

The action with $I_Q\partial_x I_Q u = 2I_Q\langle u,\partial_x\rangle I_Q - uI_Q\partial_x I_Q$ leads to

$$\begin{aligned} I_Q\partial_x I_Q u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} = {} & 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle v,\partial_x\rangle I_Q\langle u,\partial_x\rangle\mathcal{M}_{h-1} \\ & - 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle u,\partial_x\rangle I_Q\langle v,\partial_x\rangle\mathcal{M}_{h-1} \\ & - 8\left(\frac{h-1}{m-1}\right)\langle u,v\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1} \\ & - 4(h-1)uvI_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1} \\ & - 4\frac{(h-1)(m-3)}{m-1}|u|^2 I_Q\langle v,\partial_x\rangle I_Q\mathcal{M}_{h-1}. \end{aligned}$$

Next, we calculate the action with ∂_u, ∂_v and $\partial_u\partial_v$, respectively:

$$\begin{aligned} & \partial_u\left(I_Q\partial_x I_Q u\Phi_v\langle u,\partial_x\rangle\right)\mathcal{M}_{h-1} \\ & = -4\frac{(h-1)(m-3)}{m-1}\left(2uI_Q\langle v,\partial_x\rangle I_Q - mvI_Q\langle u,\partial_x\rangle I_Q\right)\mathcal{M}_{h-1} \qquad (10.15) \\ & \partial_v\left(I_Q\partial_x I_Q u\Phi_v\langle u,\partial_x\rangle\right)\mathcal{M}_{h-1} \\ & = -4\frac{(h-1)(m-3)}{m-1}\left((m-2)uI_Q\langle u,\partial_x\rangle I_Q\right)\mathcal{M}_{h-1} \qquad (10.16) \\ & \partial_u\partial_v\left(I_Q\partial_x I_Q u\Phi_v\langle u,\partial_x\rangle\right)\mathcal{M}_{h-1} \\ & = 4\frac{(h-1)(m-3)(m-2)(m+2)}{m-1}I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}. \qquad (10.17) \end{aligned}$$

Applying once more Corollary 42, we find

$$\begin{aligned} \Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} & = I_Q\partial_x I_Q u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} \\ & \quad + \frac{1}{m(m-2)}\langle\widetilde{u,v}\rangle\partial_v I_Q\partial_x I_Q\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} \\ & = I_Q\partial_x I_Q u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} \\ & \quad - 4\frac{(h-1)(m-3)}{m(m-1)}\langle\widetilde{u,v}\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1} \end{aligned}$$

with

$$\begin{aligned} \langle\widetilde{u,v}\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1} = & - (m-2)\langle u,v\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1} \\ & - (m-1)uvI_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1} \\ & - |u|^2 I_Q\langle v,\partial_x\rangle I_Q\mathcal{M}_{h-1}. \end{aligned}$$

Combining all these results, we find

$$\begin{aligned}\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} = &\ 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle v,\partial_x\rangle I_Q\langle u,\partial_x\rangle\mathcal{M}_{h-1}\\ &- 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle u,\partial_x\rangle I_Q\langle v,\partial_x\rangle\mathcal{M}_{h-1}\\ &+ 4(h-1)\langle u,v\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}\\ &+ 12\frac{(h-1)}{m}vuI_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}\\ &- 4\frac{(h-1)(m-3)}{m}|u|^2 I_Q\langle v,\partial_x\rangle I_Q\mathcal{M}_{h-1}.\end{aligned} \tag{10.18}$$

Now, we calculate $\Phi_v\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1}$. We have that

$$\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1} = 2I_Q\langle u,\partial_x\rangle I_Q\langle u,\partial_x\rangle\mathcal{M}_{h-1} - 2(h-1)|u|^2\mathcal{M}_{h-1}.$$

The action with $I_Q\partial_x I_Q\widetilde{v} = 2I_Q\widetilde{\langle v,\partial_x\rangle}I_Q - \widetilde{v}I_Q\partial_x I_Q$ on the above expression leads to

$$\begin{aligned}I_Q\partial_x I_Q\widetilde{v}\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1} = &\ 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle v,\partial_x\rangle I_Q\langle u,\partial_x\rangle\mathcal{M}_{h-1}\\ &- 4I_Q\langle u,\partial_x\rangle I_Q I_Q\langle u,\partial_x\rangle I_Q\langle v,\partial_x\rangle\mathcal{M}_{h-1}\\ &- 8(h-1)\langle u,v\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}\\ &- 12(h-1)uvI_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}\\ &- 4(h-1)|u|^2 I_Q\langle v,\partial_x\rangle I_Q\mathcal{M}_{h-1}.\end{aligned}$$

It is not difficult to calculate that the action with ∂_u, ∂_v and $\partial_u\partial_v$, respectively, leads to

$$\begin{aligned}&\partial_u\left(I_Q\partial_x I_Q\widetilde{v}\Phi_u\langle u,\partial_x\rangle\right)\mathcal{M}_{h-1}\\ &\quad= -12(h-1)\bigg(2uI_Q\langle v,\partial_x\rangle I_Q - mvI_Q\langle u,\partial_x\rangle I_Q\bigg)\mathcal{M}_{h-1}\\ &\partial_v\left(I_Q\partial_x I_Q\widetilde{v}\Phi_u\langle u,\partial_x\rangle\right)\mathcal{M}_{h-1}\\ &\quad= -12(h-1)\bigg((m-2)uI_Q\langle u,\partial_x\rangle I_Q\bigg)\mathcal{M}_{h-1}\\ &\partial_u\partial_v\left(I_Q\partial_x I_Q\widetilde{v}\Phi_u\langle u,\partial_x\rangle\right)\mathcal{M}_{h-1}\\ &\quad= 12(h-1)(m-2)(m+2)I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}.\end{aligned}$$

This is very similar to (10.15), (10.16) and (10.17), as expected.

Applying once more Corollary 42, we find

$$\begin{aligned}\Phi_v\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1} &= I_Q\partial_x I_Q\widetilde{v}\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1}\\ &\quad - 12\frac{(h-1)}{m}\langle\widetilde{u,v}\rangle I_Q\langle u,\partial_x\rangle I_Q\mathcal{M}_{h-1}\end{aligned}$$

and this leads exactly to the expression from (10.18). We can thus conclude that

$$\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} = \Phi_v\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1}.$$

Finally, we verify that, indeed,

$$\Phi_u\Phi_v : \langle u,\partial_x\rangle\mathcal{M}_{h-1} \to \mathrm{Ker}_h\mathcal{Q}_{2,1}.$$

This follows from a direct calculation, using Lemma 43, Lemma 44, Lemma 46 and the results

$$\widetilde{v}|u|^2\mathcal{M}_{h-1} = -2u\big(\langle u,v\rangle + vu\big)\mathcal{M}_{h-1}$$

and

$$\begin{aligned}&2\Big(vI_Q\langle u,\partial_x\rangle I_Q\langle u,\partial_x\rangle - uI_Q\langle u,\partial_x\rangle I_Q\langle v,\partial_x\rangle\Big)\mathcal{M}_{h-1}\\ &= \Big(\widetilde{v}I_Q\langle u,\partial_x\rangle I_Q\langle u,\partial_x\rangle + uI_Q\langle\widetilde{v,\partial_x}\rangle I_Q\langle u,\partial_x\rangle\Big)\mathcal{M}_{h-1}.\end{aligned}$$

We have

$$\begin{aligned}&\partial_x\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1}\\ &= -\frac{6(m+h-1)(m-1)}{m}u\Big(2I_Q\langle\widetilde{v,\partial_x}\rangle I_Q\langle u,\partial_x\rangle\\ &\quad - 4\frac{h-1}{m-1}\big(\langle u,v\rangle + vu\big)\Big)\mathcal{M}_{h-1}\\ &\quad - \frac{2(m+h-1)(m-3)}{m}\widetilde{v}\Big(2I_Q\langle u,\partial_x\rangle I_Q\langle u,\partial_x\rangle - 2(h-1)|u|^2\Big)\mathcal{M}_{h-1},\end{aligned}$$

whence

$$\begin{aligned}\partial_x\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} &= -\frac{6(m+h-1)(m-1)}{m}u\,\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1}\\ &\quad - \frac{2(m+h-1)(m-3)}{m}\widetilde{v}\,\Phi_u\langle u,\partial_x\rangle\mathcal{M}_{h-1}\end{aligned}$$

Hence, $\mathcal{Q}_{2,1}\Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h-1} = 0$. □

Summary

As a final note, let us verify some results in chapter 8. The kernel space, embedding factors included, is represented as follows:

$$\begin{array}{ccccc} & \Phi_v\langle u,\partial_x\rangle^2\mathcal{M}_{h+1} & & \Phi_u\Phi_v\langle u,\partial_x\rangle\mathcal{M}_{h+1} & \\ \langle u,\partial_x\rangle\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h+2,1} & & \left\{\begin{array}{c}\Phi_v\langle u,\partial_x\rangle\mathcal{S}_{h,1}\\ \Phi_u\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h,1}\end{array}\right\} & & \Phi_u\Phi_v\mathcal{S}_{h-2,1} \\ & \langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h+1,2} & & \Phi_v\mathcal{S}_{h-1,2} & \\ & & & & \\ & \langle u,\partial_x\rangle\mathcal{S}_{h+1,1,1} & & \Phi_u\mathcal{S}_{h-1,1,1} & \\ & & \mathcal{S}_{h,2,1} & & \end{array}$$

It follows from the results of this section that

$$\begin{aligned} \mathrm{Ker}_h\,\mathcal{Q}_{2,1}\cap\mathrm{Ker}\langle\widetilde{\partial_u,\partial_x}\rangle = {} & \mathcal{S}_{h,2,1}\oplus\langle u,\partial_x\rangle\mathcal{S}_{h+1,1,1}\oplus\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h+1,2} \\ & \oplus\langle u,\partial_x\rangle\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h+2,1}\oplus\Phi_v\mathcal{S}_{h-1,2} \\ & \oplus\left(a_1\Phi_v\langle u,\partial_x\rangle+a_2\Phi_u\langle\widetilde{v,\partial_x}\rangle\right)\mathcal{S}_{h,1} \\ & \oplus\Phi_v\langle u,\partial_x\rangle^2\mathcal{M}_{h+1} \end{aligned}$$

with two constants $a_1, a_2 \neq 0$ that satisfy

$$3\left(\frac{h-1}{m}\right)a_1+\left(2m+2h-2-\frac{h+1}{m}\right)a_2=0.$$

This means that

$$\mathrm{Ker}_h\,\mathcal{Q}_{2,1}\cap\mathrm{Ker}\langle\widetilde{\partial_u,\partial_x}\rangle\cong\mathcal{M}^s_{h,2,1}\oplus\mathcal{M}_{h-1,2}.$$

Furthermore,

$$\begin{aligned} \mathrm{Ker}_h\,\mathcal{Q}_{2,1}\cap\mathrm{Ker}\langle\partial_v,\partial_x\rangle = {} & \mathcal{S}_{h,2,1}\oplus\langle u,\partial_x\rangle\mathcal{S}_{h+1,1,1}\oplus\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h+1,2} \\ & \oplus\langle u,\partial_x\rangle\langle\widetilde{v,\partial_x}\rangle\mathcal{S}_{h+2,1}\oplus\Phi_u\mathcal{S}_{h-1,1,1} \\ & \oplus\left(b_1\Phi_v\langle u,\partial_x\rangle+b_2\Phi_u\langle\widetilde{v,\partial_x}\rangle\right)\mathcal{S}_{h,1} \end{aligned}$$

with two constants $b_1, b_2 \neq 0$ that satisfy

$$\left(2m + 2h - 6 + 3\frac{h-1}{m}\right) b_1 + \left(\frac{h+1}{m}\right) b_2 = 0.$$

This means that

$$\mathrm{Ker}_h\mathcal{Q}_{2,1} \cap \mathrm{Ker}\langle\partial_v, \partial_x\rangle \cong \mathcal{M}^s_{h,2,1} \oplus \mathcal{M}^s_{h-1,1,1}.$$

Finally, we have

$$\widetilde{\langle\partial_u, \partial_x\rangle}\langle\partial_v, \partial_x\rangle\mathrm{Ker}_h\mathcal{Q}_{2,1} = c_1\mathcal{S}_{h-2,1} \oplus c_2\langle u, \partial_x\rangle\mathcal{M}_{h+1} \cong \mathcal{M}_{h-2,1}$$

with

$$\begin{aligned} c_1 &= 24(m-2)(m+2)(m+h-2)(m+h-3) \\ c_2 &= \frac{6}{m}(m-3)(m+h-1)(m+2)(m+h-2). \end{aligned}$$

Remark 45. *The constants c_1 and c_2 are nonzero on condition that m satisfies $m > 3$.*

This can be written as

$$\mathrm{Ker}_{h-2}\mathcal{R}_1 \cap \mathrm{Im}\widetilde{\langle\partial_u, \partial_x\rangle} \cap \mathrm{Im}\langle\partial_v, \partial_x\rangle \cong \mathcal{M}_{h-2,1}.$$

All these results are in correspondence with the results of chapter 8.

Nederlandse samenvatting

Het onderliggend doctoraatsproefschrift handelt over de constructie van een welbepaalde hogere spin diracoperator en de studie van functietheoretische eigenschappen van deze operator. We willen benadrukken dat deze tekst niet enkel een overzicht geeft van definities, stellingen en bewijzen; hij is ook bedoeld om de lezer (en potentiële onderzoeker) een inzicht te verschaffen in het onderwerp van hogere spin diracoperatoren in een cliffordanalytisch kader. De voorbeelden en berekeningen in dit proefschrift kunnen daartoe bijdragen, hoewel die soms lang en ingewikkeld zijn.

We geven nu een gedetailleerd overzicht van de inhoud van de verschillende hoofdstukken.

Hoofdstuk 1: Inleiding

In dit inleidend hoofdstuk wordt een specifieke hogere spin diracoperator $\mathcal{Q}_{k,l}$, de hoofdrolspeler in dit proefschrift, gekaderd in de wereld der hogere spin diracoperatoren. Vervolgens wordt de opbouw van het onderliggend proefschrift geschetst, waarbij elk hoofdstuk apart onder de loep wordt genomen.

Een *hogere spin diracoperator* (zie [15, 38]) is een unieke (op een multiplicatieve constante na), elliptische, conform-invariante, eerste-orde differentiaaloperator van de vorm

$$\mathcal{D}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda)$$

met $\mathbb{V}_\lambda$ een *irreduciebele representatie* van de spingroep. Met de representatie $\mathbb{V}_\lambda$ correspondeert een *highest weight* $\lambda = (\lambda_1 + \frac{1}{2}, \ldots, \lambda_{n-1} + \frac{1}{2}, \frac{1}{2})$ waarbij $\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0$ natuurlijke getallen zijn, $n = \lfloor \frac{m}{2} \rfloor$ en m oneven. In

[73] werd aangetoond dat $\mathbb{V}_\lambda$ kan worden gerealiseerd in cliffordanalyse als een vectorruimte van polynomen in verschillende vectorvariabelen, de zogenaamde simplicaal monogenen.

Hoofdstuk 2: Basisbegrippen

Dit hoofdstuk is gewijd aan de introductie van welbekende en minder bekende eigenschappen in cliffordanalyse en representatietheorie. De laatste jaren is het duidelijk geworden dat cliffordanalyse een goede taal is om representatietheoretische resultaten te bestuderen; het omgekeerde geldt ook, want representatietheorie biedt vaak een beter inzicht in cliffordanalyse (zie [27, 9, 65, 66, 67]). Het is bijgevolg belangrijk om cliffordanalyse en representatietheorie niet los van elkaar te beschouwen.

We beginnen met de definitie van de cliffordalgebra $\mathbb{C}_m$ en enkele van zijn belangrijke deelgroepen en deelruimten, respectievelijk de spingroep $\mathrm{Spin}(m)$ en de spinorruimte $\mathbb{S}$. Vervolgens worden elementaire definities van de diracoperator ∂_x en de laplace-operator Δ_x in cliffordanalyse gegeven. De (klassieke) diracoperator ∂_x kan worden beschouwd als de veralgemening van de cauchy-riemannoperator in het complexe vlak $\mathbb{C}$. In het bijzonder geldt er dat $\partial_x^2 = -\Delta_x$, m.a.w. de cauchy-riemannoperator factoriseert de tweedimensionale laplaciaan. Nuloplossingen van ∂_x worden monogene functies genoemd en kunnen gezien worden als de veralgemening van holomorfe functies in het vlak, i.e. de nuloplossingen van de cauchy-riemannoperator. Een belangrijke rol in cliffordanalyse (zie [4, 30, 68, 42]) is weggelegd voor de vectorruimte van sferische monogenen $\mathcal{M}_h$ van graad $h \in \mathbb{N}$, i.e. h-homogene nuloplossingen van de diracoperator ∂_x. Dit is een irreducibele representatie van $\mathrm{Spin}(m)$.

In het onderdeel over representatietheorie verklaren we begrippen zoals 'irreducibele representatie' en 'highest weight' aan de hand van voorbeelden in cliffordanalyse. We focussen op representaties van de semisimpele lie-algebra's $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(3,\mathbb{C})$ en $\mathfrak{so}(m,\mathbb{C})$ (zie [40, 49, 44, 21]). Zoals eerder vermeld, kunnen irreducibele representaties van $\mathrm{Spin}(m)$ (of van zijn lie-algebra $\mathfrak{so}(m,\mathbb{C})$) gerealiseerd worden in cliffordanalyse als vectorruimten van polynomen, de zogenaamde simpliciaal harmonieken en simplicaal monogenen (zie [73]).

Tot slot introduceren we een vectorruimte van polynomen $\mathcal{M}^s_{h,k,l}$ die van bijzonder groot belang is in de constructie van de ruimte der h-homogene nuloplossingen van $\mathcal{Q}_{k,l}$, die we noteren als $\mathrm{Ker}_h\mathcal{Q}_{k,l}$. De vectorruimte $\mathcal{M}^s_{h,k,l}$ is reducibel onder de actie van de spingroep en we bepalen de decompositie in $\mathrm{Spin}(m)$-irreducibele vectorruimten.

Hoofdstuk 3: De (hogere spin) diracoperator

De klassieke diracoperator ∂_x kan worden beschouwd als een speciaal geval van de hogere spin diracoperatoren waarbij $\lambda_1 = \ldots = \lambda_{n-1} = 0$. Dit laatste betekent dat $\mathbb{V}_\lambda$ isomorf is met de spinorruimte $\mathbb{S}$. De diracoperator ∂_x is dus een endomorfisme van $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S})$ (zie [30]). In dit hoofdstuk geven we een overzicht van enkele functietheoretische eigenschappen van ∂_x, zoals de fischerdecompositie (in verschillende gedaanten), de cauchy-kowalewskaia-extensie (CK-extensie), de fundamentele oplossing en integraalformules (zie [4, 30]).

Vervolgens belichten we de surjectiviteit en conform-invariantie van hogere spin diracoperatoren zoals die werden gedefinieerd hierboven (zie ook [65, 38, 15]). We introduceren de operator $\mathcal{Q}_{k,l}$, die gedefinieerd wordt als de hogere spin diracoperator

$$\mathcal{Q}_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l});\ f(x; u, v) \mapsto \mathcal{Q}_{k,l} f(x; u, v).$$

met $\mathbb{V}_\lambda \cong \mathcal{S}_{k,l}$, i.e. de vectorruimte van de simpliciaal monogenen in twee vectorveranderlijken. Dit is een irreduciebele representatie van de spingroep met $\lambda = (k + \frac{1}{2}, l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ waarbij $k \geq l$. Naast $\mathcal{Q}_{k,l}$ geven we twee andere voorbeelden van hogere spin diracoperatoren die aan bod komen in dit proefschrift (in hoofdstukken 4 en 5).

Hoofdstuk 4: De rarita-schwingeroperator $\mathcal{R}_k$

De rarita-schwingeroperator $\mathcal{R}_k$ is de hogere spin diracoperator die inwerkt op functies met waarden in de vectorruimte $\mathbb{V}_\lambda \cong \mathcal{M}_k$ met $\lambda_1 = k$ en $\lambda_2 = \ldots = \lambda_{n-1} = 0$. In het geval $k = 1$ is de vectorruimte $\mathcal{M}_1$ een irreduciebel Spin(m)-moduul met $\lambda = (\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Het is welbekend uit de theoretische fysica dat de rarita-schwingervergelijking de relativistische veldvergelijking is van spin-$\frac{3}{2}$ fermionen; vandaar de naam 'rarita-schwingeroperator'. In [18, 19] werd deze operator bestudeerd vanuit een cliffordanalytisch standpunt en een algebraïsch meetkundig standpunt, waarbij technieken uit representatietheorie werden aangewend.

Aangezien de operator $\mathcal{Q}_{k,l}$ de directe veralgemening is van $\mathcal{R}_k$, met $\mathcal{Q}_{k,0} = \mathcal{R}_k$, zijn de eigenschappen van deze laatste operator een bron van inspiratie geweest voor de de studie van $\mathcal{Q}_{k,l}$. Echter, het geval van $\mathcal{Q}_{k,l}$ zorgt voor complicaties die nog niet zichtbaar waren voordien; deze operator is dus allerminst een triviale veralgemening van $\mathcal{R}_k$.

Hoofdstuk 5: De hogere spin diracoperator $\mathcal{Q}_{1,1}$

De operator $\mathcal{Q}_{1,1}$ kan enerzijds worden beschouwd als de eenvoudigste onder de operatoren $\mathcal{Q}_{k,l}$ met algemene $k \geq l$, en anderzijds als een speciaal geval van de hogere spin operator $\mathcal{Q}_{\lambda_j}$ die inwerkt op functies met waarden in een ruimte van spinorwaardige vormen (zie [16, 17]). In dit hoofdstuk geven we de constructie van deze operatoren in cliffordanalyse vooraleer we ons concentreren op de operator $\mathcal{Q}_{1,1}$. De resultaten van dit hoofdstuk zijn van belang voor hoofdstuk 8, waar we de ruimte $\mathrm{Ker}_h \mathcal{Q}_{k,l}$ van nuloplossingen beschrijven.

Hoofdstuk 6: Constructie van de operator $\mathcal{Q}_{k,l}$

Dit hoofdstuk begint met de formulering en het bewijs van de veralgemeende fischerdecompositie van simpliciaal harmonische functies in twee vectorveranderlijken. Deze decompositie geeft aanleiding tot de constructie van drie unieke (op een multiplicatieve constante na), conform-invariante, eerste-orde differentiaaloperatoren. De hogere spin diracoperator $\mathcal{Q}_{k,l}$ is één ervan; de andere twee zijn duale twistoroperatoren, $\mathcal{T}_1$ en $\mathcal{T}_2$, die bovendien commuteren. We geven de definitie van deze drie operatoren en berekenen hun expliciete uitdrukking in cliffordanalyse. Deze operatoren kunnen geschreven worden als $\mathcal{Q}_{k,l} = \pi_1 \partial_x$, $\mathcal{T}_1 = \pi_2 \partial_x$ en $\mathcal{T}_2 = \pi_3 \partial_x$ met π_j $(j \in \{1,2,3\})$ een projectie-operator. In het bijzonder,

$$\mathcal{Q}_{k,l} f = \left(\mathbf{1} + \frac{u\partial_u}{m+2k-2} + \frac{v\partial_v}{m+2l-4} - 2\frac{u\langle v, \partial_u \rangle \partial_v}{(m+2k-2)(m+2l-4)} \right) \partial_x f$$

met $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$.

Vervolgens bewijzen we verschillende eigenschappen van deze operatoren. Een belangrijk resultaat, met het oog op hoofdstuk 8, is bijvoorbeeld dat de twistoroperatoren nuloplossingen van $\mathcal{Q}_{k,l}$ afbeelden op nuloplossingen van respectievelijk $\mathcal{Q}_{k-1,l}$ en $\mathcal{Q}_{k,l-1}$.

Hoofdstuk 7: Resultaten in cliffordanalyse

Enkele functietheoretische eigenschappen van de operator $\mathcal{Q}_{k,l}$ worden besproken in dit hoofdstuk, zoals de veralgemeende CK-extensie, de fundamentele oplossing van $\mathcal{Q}_{k,l}$ en veralgemeende elementaire integraalstellingen. De constructie van de fundamentele oplossing steunt op een resultaat in [65] dat zegt dat nuloplossingen van een conform-invariante operator invariant zijn onder de actie van de conforme groep. Deze stelling vormde de inspiratie om de algemene vorm van fundamentele oplossing van de conform-invariante hogere spin diracoperator $\mathcal{Q}_{k,l}$ te formuleren (zie ook [70]). Het bewijs maakt onder andere gebruik van de theorie van distributies die verwant zijn aan rieszpotentialen (zie [45, 41]). Aan de hand van de fundamentele oplossing van $\mathcal{Q}_{k,l}$ kunnen we de veralgemening formuleren en bewijzen van de volgende elementaire integraalstellingen: de stelling van Stokes, de stelling van Cauchy-Pompeiu en de formule van Cauchy.

Hoofdstuk 8: Constructie van de kern van $\mathcal{Q}_{k,l}$

We hebben reeds vermeld dat de ruimte $\mathcal{M}_h$ van h-homogene nuloplossingen van ∂_x een irreduciebele representatie is van de spingroep. Als we gebruik maken van een notatie die eerder werd ingevoerd, kan dit geschreven worden als $\mathrm{Ker}_h\partial_x = \mathcal{M}_h$. Uit [18, 19] en [16, 17] weten we dat respectievelijk $\mathrm{Ker}_h\mathcal{R}_k$ $(h \geq k)$ en $\mathrm{Ker}_h\mathcal{Q}_{1,1}$ $(h > 1)$ niet meer irreduciebel zijn onder de actie van Spin(m); deze vectorruimten kunnen worden ontbonden als een directe som van Spin(m)-irreduciebele modulen die isomorfe kopieën zijn van de vectorruimte der simpliciaal monogenen van een bepaalde graad.

Net zoals in [18] voor de operator $\mathcal{R}_k$, vallen de nuloplossingen voor $\mathcal{Q}_{k,l}$ uiteen in twee types (A en B) van oplossingen. Oplossingen van het type A worden gegeven door $f \in \mathrm{Ker}_h\mathcal{Q}_{k,l}$ die voldoen aan $\partial_x f = 0$; een polynoom $f \in \mathrm{Ker}_h\mathcal{Q}_{k,l}$ is van het B-type als en slechts als $\partial_x f \neq 0$ en $\pi_1\partial_x f = 0$. In [10] wordt aangetoond dat f van het type A is als en slechts als $f \in \mathcal{M}^s_{h,k,l}$. De oplossingen van het B-type worden bepaald aan de hand van een inductieprincipe. De formulering van de inductiehypothese maakt gebruik van de theorie van stelsels van diracvergelijkingen van de vorm $\partial_x f = g$ in meerdere veranderlijken (zie [24]). In het geval van $\mathcal{Q}_{k,l}$ gaat het om een stelsel van drie dergelijke vergelijkingen. In [24] werd aangetoond dat dit stelsel oplosbaar is als er aan bepaalde compatibileitsvoorwaarden voldaan is. Deze voorwaarden geven aanleiding tot de formulering van de volgende stelling:

Theorem 30. *Voor alle $h \geq k+l$ geldt er dat*

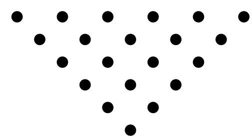
$$\mathcal{M}^s_{h,k,l} \oplus \left(\mathrm{Ker}_{h-1}\mathcal{Q}_{k-1,l} \cap \mathrm{Ker}\mathcal{T}_1\right) \oplus \left(\mathrm{Ker}_{h-1}\mathcal{Q}_{k,l-1} \cap \mathrm{Ker}\mathcal{T}_2\right)$$
$$\oplus \left(\mathrm{Ker}_{h-2}\mathcal{Q}_{k-1,l-1} \cap \mathrm{Im}\mathcal{T}_1\mathcal{T}_2\right) \hookrightarrow \mathrm{Ker}_h\mathcal{Q}_{k,l}.$$

Om te bewijzen dat het linkerlid isomorf is met het rechterlid, maken we gebruik van een dimensionale analyse en de inductiehypothese. Het resultaat is dat

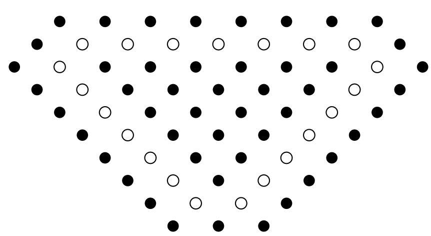
$$\mathrm{Ker}_h\,\mathcal{Q}_{k,l} \cong \bigoplus_{i=0}^{k-l}\bigoplus_{j=0}^{l}\bigoplus_{p=0}^{k-i-l+j}\bigoplus_{q=0}^{l-j}\mathcal{S}_{h-i-j+p+q,k-i-p,l-j-q}.$$

Hoofdstuk 9: Geometrie van de kern

Het doel van dit hoofdstuk is om een visuele voorstelling te vinden van de irreduciebele modulen in $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$. In het geval van $\mathrm{Ker}_h\,\mathcal{R}_k$ werden deze irreduciebele modulen (gevisualiseerd als “$\bullet$”) als volgt voorgesteld:



In het geval van $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$ herschrijven we de ingewikkelde bovenstaande som zó dat we de modulen kunnen hergroeperen. We benadrukken dat dit is een bijzonder technisch hoofdstuk is, maar het resultaat oogt zeer elegant. We vinden nog steeds driehoeksstructuren terug, maar er duiken dit keer ook zeshoeken op. Modulen in $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$, opnieuw aangeduid met “$\bullet$” en “$\circ$”, kunnen dan als volgt worden geordend:



Hoofdstuk 10: Inbeddingsfactoren

Het doel van dit hoofdstuk is om de ruimtes $\mathcal{S}_{h-i-j+p+q,k-i-p,l-j-q}$ in te bedden in $\mathrm{Ker}_h\,\mathcal{Q}_{k,l}$ aan de hand van een 'inbeddingsfactor'

$$\mu : \mathcal{S}_{h-i-j+p+q,k-i-p,l-j-q} \hookrightarrow \mathrm{Ker}_h\,\mathcal{Q}_{k,l}.$$

De inversie-operator I_Q kan gezien worden als een veralgemening van de kelvinoperator (zie e.g. [29]) en is essentieel in de constructie van de inbeddingsfactoren. Deze inversie-operator beeldt oplossingen van $\mathcal{Q}_{k,l}$ af op oplossingen van $\mathcal{Q}_{k,l}$ en is bijgevolg een ideale kandidaat voor de doelstelling van dit hoofdstuk. Een van de redenen waarom het geval van $\mathcal{Q}_{k,l}$ zo ingewikkeld is, is de aanwezigheid van irreducibiele modulen met multipliciteit groter dan één. Deze hoge multipliciteiten zorgen ervoor dat we niet langer de aanpak kunnen volgen van in [18]. We moeten het dus over een andere boeg gooien, en dit heeft geleid tot een formulering van de inbeddingsfactoren Φ_u en Φ_v zoals in Propositie 27. We hebben deze inbeddingsfactoren gecontroleerd aan de hand van drie voorbeelden: het geval van de rarita-schwingeroperator $\mathcal{R}_k$, de ruimte $\mathcal{M}^s_{h,k,l}$ en de operator $\mathcal{Q}_{2,1}$. Hoewel we ervan overtuigd zijn dat de inbeddingsfactoren correct zijn, is het niet eenvoudig om te bewijzen dat

$$\begin{aligned}\Phi_u &: \mathrm{Ker}_{h-1}\,\mathcal{Q}_{k-1,l} \to \mathrm{Ker}_h\,\mathcal{Q}_{k,l}\\ \Phi_v &: \mathrm{Ker}_{h-1}\,\mathcal{Q}_{k,l-1} \to \mathrm{Ker}_h\,\mathcal{Q}_{k,l},\end{aligned}$$

omdat de berekeningen zeer snel zeer ingewikkeld worden. Dit geeft aan dat onze methode om hogere spin diracoperatoren te onderzoeken niet de meest effectieve is. De aanpak in [72] ziet er zeer veelbelovend uit. Niettegenstaande het feit dat we dit hoofdstuk moeten afsluiten met een vermoeden in plaats van een stelling en bewijs, zijn er veel resultaten terug te vinden, die waarschijnlijk ooit kunnen worden ingezet om bepaalde patronen tussen operatoren te verklaren.

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